



Included within the medical encyclopedia of Dioscorides (fl. mid-first century A.D.) is a survey of the plant species and their therapeutic value. The drawings of ivy shown here are taken from the famous illustrated Byzantine manuscript of the work (made in 512 A.D.) and now held in the collection of ancient manuscripts of the Austrian National Library (Cod. Vindobon. Med. Gr. 1, f. 174v). This reproduction has been made from the facsimile edition of the Codex (Akademische Druck- und Verlagsanstalt, Graz, 1966-70, vol. 2), with the kind permission of the publishers and the Oesterreichische Nationalbibliothek. (For discussion, see chap. 6, sect. iv.)

The Ancient Tradition of Geometric Problems

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Preface

Within the ancient geometry, a geometric "problem" seeks the construction of a figure corresponding to a specific description. The solution to any problem requires for its completion an appeal to the constructions in other problems already solved, and in turn will be applied to the solutions of yet others. In effect, then, the corpus of solved problems forms an ordered sequence in which each problem can be reduced to those preceding. The implications of this simple conception struck me, as I was completing a paper on Apollonius' construction of the hyperbola (1980; published in *Centaurus*, 1982), for it served to unify a diverse range of geometric materials I had then been collecting for some five years.

That the ancient problem-solving effort took on such a structure is much what one would expect, given the prominence of the role of "analysis" for discovering and proving solutions; for this method seeks in each case to reduce the stated problem to others already solved. We possess one ancient work, Euclid's *Data*, which organizes the materials of elementary geometry in this manner; but there survives none which attempts the same for the more advanced field. The *Conics* of Apollonius, for instance, is by its own account ancillary to this effort, providing the essential introduction to the theory of conics through which the so-called "solid" problems might be solved; but it undertakes the actual solution of such problems only to a very limited extent. This salient omission from the extant record thus defines the project of the present work: to exploit the materials extant from Archimedes, Apollonius, Pappus and others in order to retrieve a sense of the nature and development of this ancient tradition of analysis.

The present effort is conceived as an exploratory essay, intended to reveal the opportunities which the evidence available to us provides for an interpretation of the ancient field. A definitive survey overreaches its scope, however, and may ultimately be unattainable in view of the gaps in our documentation. Sim-

ilarly, an exhaustive survey of the immense secondary literature on the history of geometric constructions could not be attempted here. The sheer bulk of this literature has made omissions inevitable, but I have endeavored to include references to those contributions which seemed to me historically and technically stimulating, as well as directly pertinent to my special objectives. If through inadvertence I have omitted discussions of worth, I apologize for that and welcome information which will prevent similar lapses in the future.

Since the field of ancient geometry, in particular, the work of Archimedes, has been an area of special interest to me for several years, the present study occasionally touches on issues discussed in previous publications of my own. Thus, I sometimes have reason to summarize positions elaborated in detail elsewhere, or to allude in passing to treatments of related issues. In such cases I have striven to avoid mere repetition of previous efforts, and the present work, but for these occasional overlaps, consists of new material.

In my interpretive efforts here I will emphasize the mathematical and historical aspects of the ancient writings, while the final chapter will include aspects of philosophical interest as well. Another aspect, that dealing with textual issues of documentation, while often overlooked in studies of this type, sometimes becomes indispensable for understanding the ancient work. I thus include commentary along these lines, when it bears on specific matters taken up here. The broader inquiry, of surveying the ancient documents and working out an account of their production and transmission, is not attempted here, but forms the project of a separate volume currently in preparation. I intend to examine in that second work a set of Greek and Arabic texts on geometric problems, thereby to reveal the general character of the scholarship which has preserved testimony of the ancient geometric field.

Grants from the American Council of Learned Societies and the Institute for Advanced Study enabled me to undertake these researches at the Institute in 1978-79, and a grant from the National Science Foundation, administered by the American Academy of Arts and Sciences, provided support for continuation of this work in 1979-80.

I have profited considerably from the criticisms and suggestions of students in the Department of Philosophy at Stanford University, in particular, Susan Hollander, Henry Mendell and David O'Connor. I wish also to thank my colleagues in the Departments of Philosophy, Classics and Mathematics for their comments, and, in particular, Sol Feferman, Halsey Royden, Hans Samelson and Patrick Suppes, for reading draft chapters, and Jody Maxmin for advice in the presentation of the plates. Ian Mueller (University of Chicago) provided extensive and valuable comments on the work in its later stages. I especially appreciate the respect and courtesy with which they all received my effort, even when my positions differed sharply from their own.

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CHAPTER 1

Sifting History from Legend

The problems of cube duplication, angle trisection, and circle quadrature have deservedly attracted major interest among scholars of ancient Greek geometry. The ancient literature on these problems constitutes one of the most richly documented fields of ancient mathematics, amounting to over two dozen different solutions, often in multiple versions and spanning the whole course of antiquity from the pre-Euclidean period through the Hellenistic, Graeco-Roman, and Byzantine periods even into the Arabic Middle Ages. Throughout, these problems were affiliated with researches in the more advanced fields of geometry.

A historical survey of these studies could thus be hoped to provide insight of value into the development of the techniques of geometry in antiquity: What were the sources of the interest in these problems? How did their study relate to prior efforts on the same or associated problems? How did the results of these studies contribute to the development of new geometric techniques for the solution of geometric problems? What were the precise specifications for the construction of solutions, and did these conditions themselves change with the introduction of new techniques and concepts? Ultimately, did the ancient geometers and philosophers view the quest for solutions as having succeeded?

Unfortunately, only a small part of this project now permits its presentation in the form of a straightforward narrative of how things happened. The extant technical literature has large gaps, so that we cannot claim to know all the solutions to these problems which the ancients worked out, nor even, in some cases, the geometers responsible for solutions which have survived. More difficult still is the assessment of motive, since technical documents rarely provide

direct insight into the reasons which lead geometers to take up specific problems and treat them in specific ways. For this we are often compelled to resort to testimonies in nonmathematical writings. But we then face new difficulties, in that their authors may not fully comprehend the technical issues and, at any rate, will have their own literary or philosophical concerns. The latter will invariably discourage them from presenting the technical materials in the clear and detailed manner we would prefer.

One of the important witnesses to the early study of the cube duplication illustrates well the hazards of using such nonmathematical sources. It is a passage from the *Demon of Socrates*, a dramatization of the political conspiracies in the Greek city-state of Thebes in 379 B.C. by a writer best known for his philosophical and biographical works, Plutarch of Chaeronea (1st-2nd century A.D.).¹ The speaker here is Simmias, one of the Theban conspirators, who is reporting on a recent visit to Egypt he made in the company of Plato:

As we traveled from Egypt, certain Delians came upon us near Caria and begged Plato, as a geometer, to solve for them an unusual oracle proposed by the god. The oracle was that there would be a respite from present ills for the Delians and the other Greeks after they had doubled the altar in Delos. But as they couldn't figure out the purpose and were faring ludicrously with the construction of the altar (for on account of their unfamiliarity with the proportion which provides the double in length, they failed to notice that when each of the four [!] sides was doubled, they were effecting an eightfold increase in the solid space), they summoned Plato for assistance in the puzzle. And he, recalling the Egyptian, said that the god was making sport with the Greeks for their neglect of education, as it were taunting us for our ignorance and demanding that we engage in geometry, and not just as a pastime. For surely it is a task not for an intellect which is inferior and perceives dully, but rather for one trained to the limit in [the constructions of] lines to take two means in proportion, by which alone the figure of a cubic body is doubled by being increased in the same way from every dimension. This [he said] Eudoxus of Cnidus or Helicon of Cyzicus would work out for them; but they shouldn't think that the god wished this, but rather that he enjoined all the Greeks to abandon war and its ills and to consort with the Muses, and by soothing the passions through reasoning and mathematics to live together profitably and without harm.²

Plutarch's chief interest here is to make the point that theoretical disciplines like mathematics and philosophy can be an effective diversion from conflict, both through their moderating influence on human aggressiveness and through the power that reasoning has for reconciling differences. This very point had been made earlier in a passage alluded to here by the phrase "recalling the Egyptian," in which an Egyptian priest had informed certain Greeks that the document they brought to him for deciphering contained an appeal to cultivate the Muses instead of war.

But what does Plutarch mean to say about the origins of this mathematical inquiry itself? At first glance, we might suppose he has in mind simply that the geometers in Plato's circle began to study the cube duplication in answer to the puzzle posed by the oracle. In effect, Plutarch would be subscribing to a form

of the "externalist" view under which technical activities find their motivation in factors outside the technical field, for instance, from political or economic causes, or, as here, from philosophical and religious ones. Indeed, passages like this have been cited by those who would maintain that ritual practices formed an important stimulus for early mathematical studies or that philosophical discourse provided the first model of abstract reasoning for the Greek geometers.³ But a closer reading reveals that Plutarch does not actually claim this, for his Plato already knows certain facts about this problem, specifically, that it is equivalent to the problem of finding the two mean proportionals. If the side of the given cube is A , and we have found two lines X , Y such that $A : X = X : Y = Y : 2A$, then by compounding ratios, we have $(A : X)^3 = (A : X)(X : Y)(Y : 2A)$, whence $A^3 : X^3 = A : 2A$ or $X^3 = 2A^3$.⁴ From other sources, in particular Eratosthenes of Cyrene (3rd century B.C.), we learn that this reduction of the cube duplication to the finding of two mean proportionals was the discovery of Hippocrates of Chios, that is, around the turn of the 4th century B.C. and thus predated by several decades the events on which Plutarch's narrative is based.⁵ Plutarch himself appears to depend on Eratosthenes for his account of the Delian oracle, for a parallel passage on the cube duplication in the *Mathematical Exposition* by Theon of Smyrna (early 2nd century A.D.) cites Eratosthenes' *Platonicus* for the same incident of Plato and the Delians with the same basic point:

And Plato told them that not for want of a double altar did the god pronounce this to the Delians, but rather to accuse and reproach the Greeks for their neglect of mathematics and their slighting of geometry.⁶

Aware of Plutarch's source, we thus do well to receive with caution the details of his account. It is clear, for instance, that the chronological association of this story with the uprising in Thebes derives entirely from Plutarch's adaptation of the older version.

Similar precautions are necessary in dealing with another passage from Plutarch, providing a sequel to the Delian story by portraying how Plato reacted to the methods of solution worked out by the geometers. This passage is inserted into Plutarch's description of Archimedes' mechanical achievements, critical for the defense of Syracuse against the Roman siege of 215-212 B.C.

This admired and much talked of instrumental art was first set in motion by those around Eudoxus and Archytas, who embroidered geometry with subtlety, but who supported by means of perceptible and instrumental models problems not permitting rigorous geometric demonstration; for instance, in connection with the problem about the two means in proportion, an element necessary for many of the geometric propositions, both of them resorted to instrumental constructions by fitting out certain means-drawing apparatuses from curved lines and segments. But Plato took offense and contended with them that they were destroying and corrupting the good of geometry, so that it was slipping away from incorporeal and intelligible things toward perceptible ones and beyond this was using bodies requiring much wearisome manufacture. In this way mechanics fell out, separated from geometry, and

for a long time it was overlooked by philosophy and became one of the military arts.⁷

Plutarch thus alleges a schism between the mechanical arts and the theoretical disciplines, bridged only by Archimedes, presumably for putting into practice his insights into pure geometry. The roots of this division Plutarch quite reasonably assigns to the idealist recommendations of the Platonic philosophy. Indeed, in Plato's *Republic*, Socrates criticizes geometers for their insensitivity to the true nature of their inquiry:

They speak in a very ludicrous and restricted manner. For they make talk about squaring and applying and adding and all, in a way as if they were engaged in practice and making all their words for the sake of practice. But surely the entire discipline is studied for the sake of knowledge... and knowledge of what always exists, but not of anything that can ever be made or destroyed (527 a-b). ... They use the visible forms and make words about them, but their thinking is not about these things, but about what these things resemble, when they make their words about the square itself and the diagonal itself, but not about the one which they draw, and the others in this way; these very things which they fashion and draw, and which have shadows and images of their own, they can use them in turn as images in their search to see those things which no one could see other than by the intellect (510 d-e).⁸

It is this Platonic sentiment, this conviction of the purity of the objects of true geometric study, which Plutarch captures in his anecdote on the cube duplication. Nevertheless, we can perceive that here too he draws from Eratosthenes as an intermediary source. For the Archimedes-commentator Eutocius of Ascalon (6th century A.D.) assigns to Eratosthenes a long account of the history of this problem, including the following passage as sequel to Plato's meeting with the Delians:

And when they [sc. the geometers in the Academy] applied themselves diligently and sought how to take the two means of two given lines, Archytas of Tarentum is said to have discovered it by means of half-cylinders, while Eudoxus [did so] by means of the so-called curved lines. But it happened that all of them effected this apodictically, but couldn't construct it physically or put it into service, save to some small degree Menaechnus, and that with difficulty.⁹

Here, as in Plutarch's version, the geometers busy themselves under the watch of Plato, Archytas coming up with one solution, Eudoxus with another. Yet there is a profound discrepancy: Eratosthenes deems their solutions to be too abstract and impracticable, while Plutarch portrays them as overly mechanical and too little abstract. This kind of discrepancy is one form of evidence adduced by some scholars to discredit Eutocius' assignment of this writing to Eratosthenes.¹⁰ We will examine this argument later. For now, it is enough to note that the passages can be reconciled in a straightforward way. Two different writings by Eratosthenes are at issue here: the document on the cube duplication and the *Platonicus*; it is the latter which Plutarch exploits as his source. The *Platonicus* was presumably a fictional essay, so that its strongly antimetaphysical position here could have resulted from Eratosthenes' adaptation of the other,

more plainly historical account, where the early solutions had no pronounced mechanical element.

On the relative value of the mechanical and the theoretical elements in geometry, it is the attitude of the geometers themselves we would most like to know. The difficulties in learning this from the passages just cited are evident, since the glosses by Plutarch, Eratosthenes, and the 4th-century Platonists must first be separated out before the underlying mathematical views can be discerned. Although Plato assigned a prominent role to mathematical studies within his general program of philosophy, that in itself does not qualify him or his disciples to speak for the technical researchers on the issues which concerned them. Indeed, passages like those cited from the *Republic* suggest that Plato and other philosophers might set themselves apart as *critics* of the technical field. But if we might try to infer from such critiques that the geometers insisted on a this-worldly orientation in their studies, we have the variant testimony of Eratosthenes (in the account of the cube duplication) that the early studies were cast in an abstract manner. The mathematical texts preserved by Eutocius on the methods used by Archytas and Menaechnus bear Eratosthenes out on this point.¹¹ Yet Eratosthenes' own solution is accompanied by the detailed account of an actual physical device built on its design, and this overtly mechanical element is found in the methods used by Nicomedes and Diocles and others around the same time (late 3rd century B.C.).¹² Evidently, views on the importance of mechanical approaches varied over the course of antiquity and might sometimes be a matter merely of personal preference. In this lack of unanimity on issues of methodological commitment, the ancient field of mathematics would conform to a pattern familiar in more recent periods in the history of mathematics.¹³

These reflections thus reveal three major recommendations bearing on the present study of the ancient problem-solving efforts. First, we must attempt to establish the simple chronological sequence of the various solutions; only thus can one grasp for each period the complex of problems, methods, and aims which set the directions of geometric research at that time. On this basis one may hope to discern changes over the longer term in the character of the field of research. Typically, treatments of these materials have subscribed to a division by specific problems or by solving methods. Precedents can already be cited from antiquity in the compendium of solutions to the cube duplication presented by Eutocius in his commentary on Archimedes' *Sphere and Cylinder*, in the sections on cube duplication and angle trisection in Books III and IV of Pappus' *Collection*, and presumably also in the survey of circle quadrature in Sporus' *Keria*, and in the surveys of special curves in the encyclopedias of Menelaus and Geminus.¹⁴ Among modern discussions, one is most likely to consult the extended chapter on "Special Problems" in T. L. Heath's *History of Greek Mathematics*, the detailed articles on cube duplication, angle division, and *neusis* constructions by R. Böker in the *Pauly Wissowa Real-Encyclopädie*, the long paper on ancient studies of curved lines and surfaces by P. Tannery, and the massive compendium on the history and properties of special plane curves by G. Loria.¹⁵ Without questioning the great value of these accounts, as well as

many others too numerous to cite here, nevertheless believe that their principle of organization implicitly commits them to the view that the ancients likewise conceived of the study of these particular curves or these particular problems as distinct subfields. That might hold for an encyclopedist like Geminus, or even for commentators like Pappus or Eutocius, but not, it seems to me, for the original investigators like Archimedes and Apollonius and their contemporaries. By contrast, if we adopt a chronological scheme, we can suspend commitment on the ancient conception of the status of these problems and the various solving techniques, so that this conception will be allowed to emerge.

The "simple" chronological questions are in fact often quite difficult to determine, as we shall find. The restriction to the three "classical" problems is of course artificial itself and will not be adhered to strictly. Its obvious advantage lies in the ample documentary coverage of these problems, touching, in some form, on each of the major periods of the ancient geometry. This raises the question of why the three problems continued to attract attention, even after many different solving methods had been worked out. An important key to the answer is found in the changes affecting the general field of research from one period to the next. As new techniques were introduced, their applicability to the older problems would naturally come up for consideration: for instance, a special curve generated in the context of a method for the angle trisection might be recognized to have properties relevant for an alternative solution of the cube duplication as well.¹⁶ It is thus clear that one cannot adequately comprehend the approaches adopted in the cases of the individual problems without full regard to the more general environment of research. This presents not a difficulty, however, but an opportunity. For the documentation extant representing the more advanced researches at various times is often quite sparse. One may thus view the special problems as a cross section, suitable for filling out one's portrait of the wider field. This investigation will occupy the next six chapters of the present study (Chapters 2-7), moving from the pre-Euclidean period through the generations of Euclid and Archimedes to the time of Apollonius.

A second recommendation is to devote separate treatment to several important metamathematical issues: how the ancients divided the geometric field according to the types of problems and solving methods; what they viewed the special role of problems to be, especially in relation to that of theorems, and how they associated these with the important methods of analysis and synthesis; what conditions they imposed on the techniques admissible for the solution of problems, and whether they judged that satisfactory solutions for the three special problems had actually been found. Historical accounts of these materials almost invariably assume at the outset that the ancients had at a very early time distinguished compass-and-straightedge constructions from the others and were continually urged on in the quixotic search for constructions of this type for the three special problems.¹⁷ As we now know, through results obtained in the 19th century, the ancients were in a position neither to produce such solutions nor even to show, save through persistent failure, that they could not be produced. For the latter is essentially an algebraic question not amenable to investigation

under the geometric methods of the ancients.¹⁸ I believe, however, that we cannot assume that the ancient views on these metamathematical issues were just the ones familiar within the modern field, or that even if certain geometers at certain times held views somewhat like those now familiar, all other geometers at all other times necessarily held the same views. Surely our goal must be to discover what the ancient views were by means of a consideration of the relevant historical evidence. This is the project undertaken in Chapter 8. However, as our look at Plutarch's passages on the cube duplication suggests, the search for answers will be far from straightforward. Most of the passages we may consult come from later authors whose primary interest is in the explanation of methodological observations made by philosophers, most notably Plato and Aristotle. For instance, the neo-Platonist element pervades the important Euclid commentary by Proclus (5th century A.D.), while much of our evidence must come in the form of interpretations of specific passages from Aristotle by Alexander of Aphrodisias, Themistius, Philoponus, Simplicius, and others. Recall that our passages from Plutarch derive from his discussions of Plato, and the same is true for his source, Eratosthenes. Even with writers who one might suppose are closer to the actual mathematical tradition, such as Pappus, one can readily detect the influence of certain philosophical commitments on their mathematical pronouncements.

This aspect of the commentators helps us to understand an emphasis commonly but, I believe, mistakenly made in discussions about the ancient geometry. One seems typically to assume that metamathematical concerns were the effective motivating force underlying the efforts of geometers, for instance, that the hallmark of their tradition was their organization of geometric findings into tight structures of deductive reasoning, as if their primary ambition was the production of treatises like the *Elements* of Euclid and the *Conics* of Apollonius. But this surely cannot be correct. The writing of textbooks is the *end* of mathematical research only in the sense that death is the end of life: it is the last term in a sequence, but not the purposive element which explains why one engages in the activity of progressing through the steps in this sequence.¹⁹ As far as the ancient geometers are concerned, the goals underlying the activity of research are in most cases a matter for surmise. In general, I will find most convincing the "internalist" position that technical research is directed toward the solution of problems arising from previous and current research efforts. In effect, the formulation of problems keeps a field open by guiding new research. To be sure, the goal to rigorize the proofs of known results can stimulate one form of research; the most notable example of this among the ancients may be found in the mathematical insights of Eudoxus on which were built the formal theories of proportion and limits.²⁰ But with the exception of Eudoxus, the context of the interesting mathematical insights by the ancient geometers lies not in such formal concerns, but rather in the investigation of problems. In this respect, I believe that an intuition founded on the most recent period of mathematical history can be misleading, for metamathematical inquiries have crystallized into a recognized subfield, the study of foundations, commanding expertise of an entirely math-

ematical kind. But this did not happen in antiquity. If, for instance, the deductive project embodied in Euclid's *Elements* entails subtle questions of axiomatics, the ancients seem to have rest content with Euclid's formulation and with the Aristotelian theory to which it loosely relates, for both remain the focus of later discussions of foundational issues.²¹

The third recommendation is to take up the textual issues bearing on our use of the writings of authors from later antiquity. Their editorial efforts have preserved for us most of what now survives of the ancient geometric tradition. In some instances, they produced the extant editions of the treatises of the great geometers like Euclid, Archimedes, and Apollonius; in others, they compiled miscellanies of geometric results drawn from works now lost. It is thus extremely important for us to gain a sense of how these writers handled texts: Where did they obtain their texts? What changes did they make in them? What were their purposes in selecting and presenting these texts? In this regard, the texts on the three special problems, in particular, that of the cube duplication, provide an ideal instrument, since they survive in multiple versions from several writers. It is remarkable that so little along these lines has been attempted up until now. Even those prominent scholars, J. L. Heiberg and P. Tannery, for whom philosophical issues were a particular concern, did not much explore those questions which the comparisons of these texts might elucidate. It is as if the later writers were of no intrinsic interest to them, and that once one of several related texts could be accepted as the "best," the others might be disregarded.²² In a sequel following the present volume an attempt will be made to display the interrelations among the surviving texts of these solutions. One can well anticipate that these writers are not highly original in their mathematical contributions. But the degree to which they depend on sources, not only for mathematical techniques and results, but even for the very wording of the texts they present, I believe, will be surprising even for those who have made a study of the ancient geometry. This dependence applies as well to Pappus as to the other commentators, despite the diversity and advanced level of the materials in his *Collection*. Further, the pattern of textual methods will be seen to carry over among Arabic writers of the 9th and 10th centuries A.D. The discussion of several texts on the angle trisection and cube duplication from Arabic authors, in comparison with selected items from the earlier Greek tradition, will occupy a part of the sequel and its Appendices. There I will attempt to show that many of these Arabic texts were produced as translations, or at least close paraphrases, of the Greek, even in certain cases where one might have supposed they were original Arabic writings or reworkings of Arabic prototypes. Thus, the derivative character both of this part of the Arabic mathematical tradition as well as of the Greek editorial tradition from the Graeco-Roman and Byzantine periods becomes clear and suggests that one might hope to exploit them in new ways for gaining insight into the earlier creative phases of Greek geometry. This last is of course a project extending well beyond the scope of the present study.

In this survey of the ancient problem-solving efforts, attention to matters of textual evidence will predominate in Chapter 8 and the sequel study. Elsewhere,

the essential mathematical aspects of the different technical contributions will provide the principal instrument for revealing their historical relations. A difficulty here is that the ancient writers (especially those in the later editorial tradition) preferred the synthetic mode of exposition in their formal treatments of geometry. That is, one derives a claimed result (e.g., theorem or construction) as the conclusion of a deductive chain of reasoning which starts from the given terms of the theorem and otherwise utilizes only axioms and other first principles or previously proven theorems and constructions. To mathematicians of the 17th century, indeed, even to some of the ancients, this format was notorious for obscuring the essential line of thought, particularly in the cases of more complicated results.²³ But when the discovery of solutions for geometric problems was at issue, the ancients exploited an alternative method: that of *analysis*. In the case of a problem of construction, for instance, one here posits the desired figure as having already been effected and then deduces properties of this figure until an element of it emerges which is known from prior results to be constructible. The formal *synthesis* would then begin from these constructible terms and proceed through a deductive sequence in approximately the reverse order to that of the analysis until the desired construction has been produced. Since the analytic method can be applied fruitfully even where the solution of the problem is not yet known, it is especially well suited for the purposes of geometric research. If the ancients sometimes followed this analytic method in their presentations of known solutions as well, this would doubtless owe to the appropriateness of exposing learners to the use of this method in preparation for their own research efforts later. It also has advantages for exegesis, seeing that the analysis of a problem exposes in a natural and well motivated way the rationale behind each step of the construction. By contrast, when only the synthesis is given, these steps often appear arbitrary and without clear motivation until much later in the proof. Thus, in my accounts I will prefer analytic presentations to synthetic ones. In cases where only the synthesis is extant, I will fashion the analysis corresponding to it. Given the close parallelism between these two parts of the treatment of problems, one can recover such analyses with virtually absolute confidence. In doing this, moreover, one can sometimes discern reasons for otherwise puzzling features in the extant syntheses. Nevertheless, the reader should be aware that this is an interpretive element in the account which follows. My intent is not to paraphrase or otherwise reproduce what is already available in the extant primary literature, but rather to bring forward the essential, usually simple, geometric idea underlying each result and thus to provide an appropriate introduction for further investigation of that literature.

A few terms admitting a variety of meanings in common usage are so prominent throughout this discussion of ancient geometry that I do not dare employ them in any sense other than their formal mathematical one. This applies to "analysis" and "synthesis," for instance, which will here signify only those special geometric methods mentioned in the preceding paragraph. Similarly, the term "problem" will refer only to the geometric form of a problem, that is, to the type of proposition which seeks the construction of a figure in answer to

certain specified conditions or the determination of certain properties of a specified figure. It will never be a mere synonym of "difficulty," for in fact a geometric problem need not be difficult at all. Its essence lies in its being a "project" (from *prohallein*, "to cast forward"), that is, a demand to do something not yet done or to discover something not yet known. Another such term is "prove" along with its cognate noun "proof" and the synonyms "demonstrate" and "demonstration," respectively. Here these refer only to those deductive forms of reasoning employed in mathematics and logic. I will forego their use in metamathematical and historical contexts, where more flexible terms like "show," "reveal," "make evident," and so on, will be used in their stead. The serious student of ancient mathematics knows well that the speculative component is necessarily very high. Virtually all the interesting questions we might care to investigate are beyond the range of definite answers, and the same applies even in the instances of many quite trivial matters. It is certain, for instance, that Apollonius wrote the *Conics*; but the question as to how much of the extant text is by Apollonius himself, rather than by his editor Eutocius, has not yet received the attention it demands. In view of this, one's assertions about these materials must always be received within a range of plausibilities and probabilities. A phrase like "one thus sees . . ." signifies a claim of a rather high degree of likelihood, whereas one like "it may well be . . ." signifies a low one. An oddity of our language is that ostensible intensifiers like "surely," "certainly," and "doubtless" actually indicate quite the opposite of certitude, by signalling a situation within which doubts *can* be raised; their force is largely suasive, but will at least communicate to the reader the nature of my spiritual commitments to my claims. If I sometimes lapse into incautious use of the indicative mood, the reader should be forewarned to supply the needed qualifications.

In the matter of my interpretive method, I subscribe, however imperfectly, to three simple ideals; to consider all the available elements pertinent to a given question; to advocate that view, among all the viable contenders, which best conforms with the general character of the ancient geometric tradition and with the most straightforward conception of the mathematical field in relation to other disciplines; to prefer that view among viable alternatives which most enhances the interest value of the extant documents. Applying these principles can often be a delicate matter. In the case of the first, the mere survival of a document need not compel us to believe what it says. If two commentators happen to disagree on a certain point, we must of course try to determine which (if either) is correct and to explain why there is a discrepancy. By the same token, if the statement of only one commentator is known, that need not be true. Here the second principle assists in ascertaining whether a testimony might be suspect. If, for instance, Pappus asserts something about the geometric field which flatly disagrees with what we find in the technical literature, surely we would not hypothesize a lost corpus of writings merely to make his view correct, but rather, we would admit his error and try to figure out why he made it.²⁴ One can hope eventually to gain a sense of the intrinsic credibility of the various authorities

in their various contexts. One learns quickly, for instance, that Archimedes commits no technical errors and that any assumption to the contrary on our part would be merely presumptuous.

The second principle serves to introduce an assumption of economy in interpretations. It is of course possible that ancient mathematical thought differed from that of other periods in certain striking and fundamental respects. But I believe we must be forced to such conclusions, rather than concur in them carelessly or even as a matter of principle. It seems to me, for instance, that the technical activity of problem solving in mathematics and other fields is never particularly sensitive to external factors, like the opinions of one or another school of philosophers. To be sure, the line of division may be difficult to draw sharply in some cases, and what start as external considerations might sometimes, through a very subtle process, become part of the internal conditions of research activity. But I do not imagine the ancient geometers as constantly looking over their shoulder. It is surely absurd to suppose that Platonist notions of the perfection of circles induced mathematicians to restrict their solving techniques to compasses; or even that Zeno's paradoxes of motion instilled in Eudoxus a fear of the infinite. The notion that ancient mathematics was somehow a vast exercise in dialectical philosophy must miss a very important point: that geometry is rooted in an essentially practical enterprise of problem solving.²⁵

As for the third principle, I believe that scholars have a responsibility to keep alive the interest in the documents they examine. It seems to me quite wrong to *assume* that a certain document is a forgery, unless the reasons are compelling, for that will merely guarantee that no one takes it seriously thereafter.²⁶ Again, it seems to me wrong to *assume* that a certain mathematical writer is incompetent, unless we are driven to that view on clear grounds. On the other hand, it is equally important to avoid romantic notions about the expertise of our authorities in the face of clearly contradictory indications. The stature of Pappus as an outstanding mathematical intellect, for instance, has prevented the intelligent use of his writing as a *source* for the earlier geometric tradition.²⁷ Again, if one begins to doubt that Aristotle himself produced the extraordinary mathematical reasonings in the *Meteorologica*, this work might eventually be returned to the mainstream of thinking about ancient mathematical science.²⁸

It is perhaps in the interest of scholarly precision to approach general theses about the nature of a discipline with scepticism. But most of the major treatments of the ancient geometry, such as that of Heath, have taken that wise counsel too far. They set before us what purports to be a factual narrative of the ancient field, but which is hardly other than a *mélange* of details without strong signs of interrelation, or even of internal consistency. The most exciting aspect of this material, in my view, is that a closer examination reveals the general lines of a coherent development, and that the extant record, despite its incomplete state, actually makes sense as the remnant of an extraordinary movement in thought whose basic outline is discernible. If the reader remains unconvinced of the details of the argument in the following study, yet nevertheless comes to appreciate the unsatisfactory state of current scholarship and to perceive the pos-

sibility of achieving a coherent view of this movement, then my effort shall have attained its principal objective.

NOTES TO CHAPTER 1

¹ One may consult the *Oxford Classical Dictionary* (2nd ed., Oxford, 1970) for a brief account and bibliography of the life and work of Plutarch.

² *Moralia*, 579 a-d; cf. the text, translation and annotations on this and related passages by P. H. de Lacy and B. Einarson, *Plutarch's Moralia* (Cambridge, Mass./London: Loeb Classical Library), VII, 1959, pp. 396-399.

³ The ritual background to mathematics is a prominent concern in several articles by A. Seidenberg: cf. his "Ritual Origin of Geometry," *Archive for History of Exact Sciences*, 1963, pp. 488-527 and his "Origin of Mathematics," *Archive for History of Exact Sciences*, 1976, 18, pp. 301-342. Some of these ideas have been taken up and extended by B. L. van der Waerden, "Pre-Babylonian Mathematics," *Archive for History of Exact Sciences*, 1980, 23, pp. 1-46. The thesis of a philosophic background to pre-Euclidean mathematics, especially in the form of Eleatic influences on the Pythagoreans, has been championed by A. Szabó; cf. *Anfänge der griechischen Mathematik*, Budapest and Munich/Vienna, 1969 (translated as *The Beginnings of Greek Mathematics*, Dordrecht, 1978). I have presented a detailed critique of Szabó's views in "On the Early History of Axiomatics," in *Pisa Conference (1978) Proceedings*, ed. J. Hintikka et al., Dordrecht, 1981, I, pp. 145-186.

⁴ The restriction to given lines A , $2A$ is of course inessential; one may in general posit lines A , B , in any given ratio and obtain $A' : A'' = A : B$, as the ancient treatments of the problem are well aware.

⁵ Cf. Eratosthenes, as cited by Eutocius in *Archimedis Opera*, ed. J. L. Heiberg, III, p. 88; cf., also, Proclus, *In Euclidem*, ed. G. Friedlein, Leipzig: Teubner, 1873, p. 213. Hippocrates' work is discussed in Chapter 2.

⁶ Theon, *Expositio rerum mathematicarum*, ed. E. Hiller, Leipzig: Teubner, 1878, p. 2.

⁷ Plutarch, *Lives: Marcellus*, xiv, 5-6. An alternative version of this anecdote, mentioning the geometers Archytas, Eudoxus, and Menaechnus, appears in Plutarch's "Why Plato said that god always geometrizes" (*Moralia*, 718 e-f), one of several discussions on topics related to Plato's *Timaeus*. Certain theological overtones present in this version of the Delian story must be due to Plutarch's elaborations of his source narrative, just as in the *Marcellus* passage the concern to link this story with historical trends in geometry and mechanics must also be his.

⁸ One may compare the text, translation and comments by P. Shorey in *Plato: The Republic* (Cambridge, Mass./London: Loeb Classical Library), 1935, II, pp. 170f, 1121.

⁹ *Archimedis Opera*, ed. J. L. Heiberg, III, p. 90.

¹⁰ Notably, von Wilamowitz; cf. Chapter 2.

¹¹ *Archimedis Opera*, III, pp. 78-88; these are discussed in Chapter 3.

¹² Cf. Chapter 6.

¹³ Consider the following bold-sounding claim on the "practical purpose of geometric investigations":

The graphical solution of problems of construction forms a major goal of geometry. Even where this goal seems to a certain degree to have been forgotten through the advancing of theoretical investigation, its guiding influence on this investigation is quite easily perceived.

The statement is made by the Italian geometer, F. Enriques, in the article concluding the anthology of contributions by him and collaborators on advanced questions pertaining to elementary geometry, *Questioni riguardanti la geometria elementare* (2nd ed., 1907; I have followed the second German edition, 1923, II, p. 327).

By contrast, another mathematician has this to say about the ancient studies on the conic sections:

I believe that one did not in general effect practically this construction [of the cubic root] via conic sections, but that rather it was a theoretical means—as it can also be for us—a means for the extension of understanding [*Erkenntnis*].

(From H. G. Zeuthen, "Die geometrische Construction als 'Existenzbeweis' in der antiken Geometrie," *Mathematische Annalen*, 1896, 47, pp. 222f; this essay will be discussed further in Chapter 8.) One may expect that the same division of opinion on the relative roles of practical constructions and theoretical investigations would still mark modern discussions on the nature of mathematics.

¹⁴ Eutocius, in *Archimedis Opera*, III, pp. 54-106. Pappus, *Collection*, ed. Hultsch, I, pp. 54-68, 270-284. On Menelaus, Geminus, and Sporus, see T. Heath, *History of Greek Mathematics*, I, pp. 226, 229f; II, pp. 222-226, 260f. I will propose further views in the sequel.

¹⁵ Heath, *History of Greek Mathematics*, Oxford, 1921, I, Ch. 7. Böker, "Winkel und Kreisteilung," *Pauly Wissowa*, Ser. II, 9., 1961, col. 127-150; "Würfelverdopplung," *ibid.*, col. 1193-1223; "Neusis," *Pauly Wissowa*, Suppl. IX, 1962, col. 415-461. *Neuses* are familiar in the form of sliding marked ruler constructions; see Chapters 2, 5, and 6. Tannery, "Pour l'histoire des lignes et surfaces courbes dans l'antiquité," *Bull. Sci. Math. Astr.*, 1883, 18, 278-291 and 1884, 19, 19-30, 101-112. Loria, *Ebene Kurven: Theorie und Geschichte*, 2nd ed., Leipzig/Berlin, 1910 (1st Italian ed. 1902; 2nd Italian ed. 1909).

¹⁶ Cf. the discussion of Nicomedes' conchoid in Chapter 6.

¹⁷ Cf. Heath's treatment in *History*, I, pp. 218-220. A distinguished exception to this tendency is the long study by A. D. Steele, "Ueber die Rolle von Zirkel und Lineal in der griechischen Mathematik," *Quellen und Studien*, 1936, 3, pp. 287-369. Steele blends a careful examination of the ancient sources with a good grasp of the modern algebraic view of these issues; he argues persuasively that the restriction to compass and straightedge was only rarely the explicit condition of the ancient study of geometric problems (cf. Chapter 8).

¹⁸ For a discussion of the reducibility of third-order problems together with a detailed survey of methods of cube duplication and angle trisection, one may consult the article by A. Conti in F. Enriques, *Fragen der Elementargeometrie*, II (cited in note 13 above); for a discussion of the transcendental nature of π , also with historical background, one may consult B. Calo in the same book. In the anthology *Monographs on Topics of Modern Mathematics*, ed. J. W. A. Young, 1911 (Dover, repr., 1955) one may find an article on "Constructions with ruler and compass" by L. E. Dickson, and another on the "History and transcendence of π " by D. E. Smith.

¹⁹ The pun is possible in Greek, where the word *telos* and derivatives carry the various senses of "end," "end result," and "goal"; cf. Aristotle, *Physics*, II, 2, 194 a 30.

²⁰ On Eudoxus, see Chapter 3.

²¹ The ancient field of the philosophy of mathematics is richer than my statement suggests. But that study seems primarily to have engaged philosophers, interested in accommodating mathematical questions within their own favored philosophical outlooks, like Platonism, Epicureanism, or Scepticism. The participation of mathematicians within this field appears to have been slight, and its influence on their researches slighter still. This ancient philosophical field is the subject of recent papers and a monograph in progress by I. Mueller.

²² We have already alluded to an instance of this attitude, in the treatment of the two versions of the Delian story surviving from Eratosthenes.

²³ See, for instance, Carpus' remarks on the nature of geometric problems, cited in Chapter 8. Dissatisfaction with the ancient formal style led several 17th-century geometers to reconsider questions of method and notation; see, in particular, Descartes, *Géométrie*. The method of analysis thus served to stimulate the development of new algebraic approaches in geometry; cf. M. S. Mahoney, *The Mathematical Career of Pierre de Fermat*, Princeton University Press, 1973, Ch. II.1, III, V.

²⁴ This example is not hypothetical, cf. Chapter 8.

²⁵ See the view expressed by Enriques, cited in note 13 above.

²⁶ This is clearly the effect which von Wilamowitz' rejection of the Eratosthenes document has had, discouraging use of its potentially significant testimony to the early history of cube duplication; cf. note 10 above and Chapter 2.

²⁷ Pappus' repute both as a mathematician and as a spokesman on the nature of mathematics is an issue in Chapter 8 and will be further examined in the sequel volume.

²⁸ See Chapter 4.

CHAPTER 2

Beginnings and Early Efforts

When did geometers first initiate the quest for solutions of geometric problems? One seems to find some hints toward a form of answer from certain discussions by Proclus, who, as leader of the Academy in Athens in the 5th century A.D., included guidance in the study of elementary geometry as part of his teachings in the Platonic philosophy. His commentary on Euclid's Book I is a rich source not only of ancient views on the philosophical aspects relating to mathematics, but also on details of its historical development. From Proclus we receive several items on pre-Euclidean efforts derived from the history of geometry compiled by Aristotle's disciple Eudemus of Rhodes toward the close of the 4th century B.C. Two of these relate to problems of construction: 1, 12 (how to draw the perpendicular to a given line through a given point not on that line) and 1, 23 (how to draw an angle equal to a given angle at a given point on a given line). Reportedly, these problems were first solved by Oenopides of Chios, a geometer and astronomer, elsewhere identified as an associate of Hippocrates of Chios and hence datable to after the middle of the 5th century B.C.¹

This witness to the work of Oenopides has led scholars to wonder how it could be that such elementary efforts emerged so late. Assuming to the contrary that they must have been known much earlier, they infer that it was the specific form of the solutions, as found in Euclid's treatment, which Oenopides originated, so that he would become the one responsible for the formal project of effecting geometric constructions within the restriction of employing compass and straightedge alone, as is characteristic of Euclid's method in Book I. Presumably, then, Oenopides had access to a diverse range of ways for solving

these and other problems, and set about the task of regularizing and classifying problems according to the means of construction adopted.²

If such a bald statement of this thesis seems unsympathetic, a more subtle representation would merely camouflage its intrinsic implausibility. To be sure, much older precedents for the sort of geometric construction assigned to Oenopides can be detected. For instance, Babylonian tablets from the mid-2nd millennium B.C. display figures of circles and squares drawn with the assistance of instruments, while the Egyptians' use of cords in their practice of geometric mensuration earned for them the name "rope-stretchers" among the later Greeks.³ But the type of inquiry supposedly engaged in by Oenopides is of an entirely different order of formal sophistication: to effect constructions on the consciously formal restriction to a specified set of means. It is no obvious matter to explain how and why such a distinctly formal move in geometry should appear so early. Attempts to account for it within the context of an alleged formalization of the general geometric field through linkage with pre-Socratic philosophical developments, as with the 5th-century Eleatics and Pythagoreans, are fraught with difficulties of their own.⁴

In the case of Oenopides, such doubts are magnified upon closer consideration of the details communicated by Proclus; specifically, that Oenopides introduced his construction of the perpendicular for its utility in astronomy; and that for the phrase "at right angles" he employed the term "gnomon-wise" (*kata gnōmona*), that is, with reference to the "gnomon" or pointer of a sundial. It thus seems evident that Eudemos, Proclus' source, has gleaned his information from a treatment of astronomical constructions known under the name of Oenopides.⁵ But then it hardly follows that these were the earliest constructions of these problems, even under the special manner of construction they employ, but only that these were the earliest such instances of them known to Eudemos. More importantly, Oenopides would surely not have been occupied in a consciously formal geometric effort, for the astronomical context indicates that he was showing how to arrange the construction of astronomical devices, like sundials, via appropriate instruments of construction. The latter surely included compasses and rulers, but we have no reason to suppose that he rejected the use of others. Such devices as set squares, forms of compasses and sectors, angle-measuring devices, plummets, and sliding marked rulers were all in the repertory of techniques available to geometric practitioners in antiquity, and attested in the mathematical literature from the decades before and after Euclid.⁶

It is imperative, I maintain, to raise these doubts about the formal nature of the work of Oenopides and his contemporaries. While we might consider it obvious and natural to classify constructions according to the means employed and to assign privileged status to those demanding only compass and straightedge for their execution, our intuitions in such matters are thoroughly conditioned through knowledge and adoption of the objectives of the formal geometric tradition, advanced primarily through the works of Euclid and Apollonius. But in fact this formal restriction on the treatment of problems in itself betokens attainment of a sophisticated theoretical level. Thus, a historical inquiry into the

study of problems must recognize the sophistication implicit in this move and ask about the time, manner, and motive of its introduction. Most treatments of ancient geometry, and specifically those dealing with the history of problem solving, assume this formal motive at the outset.⁷ This assumption, I believe, presents serious obstacles against gaining insight into the development of the ancient studies as they actually happened. Let us then set aside that assumption. Through the examination of the evidence of the early work, specifically on the two problems of cube duplication and circle quadrature, we may hope to discover the roots of the ancient field of problem solving.

THE DUPLICATION OF THE CUBE

We possess accounts of the early stages of the study of the cube duplication in two reports derived from Eratosthenes of Cyrene, a prominent man of science and letters from the latter part of the 3rd century B.C. One of these is the fragment from a scene in his dialogue, the *Platonicus*, and is preserved by Theon of Smyrna and Plutarch in writings from the 2nd century A.D.⁸ The other takes the form of a letter addressed by Eratosthenes to his royal patron Ptolemy III Euergetes in association with a transcript of the description of an actual model of his device, the "mesolabe" ("mean-taker"), for the mechanical solution of this problem and the text of an epigram dedicating it in the temple of the Ptolemies. The latter account is preserved as one of eleven texts on this problem compiled by Eutocius of Ascalon, the 6th-century-A.D. commentator on Archimedes' work.⁹ The value of such a document for examining the early history of these efforts would seem evident; but it has not been exploited by historians, owing to their general acceptance of the case challenging its authenticity.¹⁰ It is thus important first to consider whether that case is to be sustained.

Eutocius' version runs to five full pages in the standard edition by J. L. Heiberg. We may divide it into five sections: (i) A historical introduction cites a scene from a tragedy about Minos by an unnamed dramatist and a legend telling of Plato's involvement with the oracle of Delos as precedents for an interest in the problem of doubling the cube. Specific note is made of efforts by Hippocrates of Chios and by three associates of Plato: Archytas, Eudoxus, and Menaechnmus. (ii) Eratosthenes' mechanism for finding means is cited and praised for its greater practicality in comparison with the earlier methods. Several contexts of potential application are enumerated, among them ship building and military engineering, by way of illustrating the utility of the device. (iii) A full account of the geometric theory of the device is given. (iv) This is followed by a description of certain physical details pertaining to the actual construction of a working model. (v) The text closes with a description of the votive monument raised by Eratosthenes to commemorate its invention: this consisted of a bronze exemplar of the "mesolabe" set atop a pillar with an explanatory inscription engraved below. A full transcript of the latter is given, including (a) a brief account of the geometric theory of the device, parallel to that in (iii), and (b) an epigram of eighteen verses singing the praises of its virtues in contrast to the

same prior efforts named in (i), and finally dedicating it to Ptolemy and his son.¹¹

Since the penetrating examination of this document by U. von Wilamowitz-Moellendorff in 1894, the authenticity of the closing epigram and of the descriptive text just preceding it [(v-a) and (v-b)] have been granted.¹² The epigram is too expertly crafted, too fine a specimen of the difficult form of Hellenistic elegiac couplets, he argued, to be so easily dismissed as a forgery the way critics before him had contended. The descriptive text on the model of the device would raise no suspicions (save for an occasional trivial interpolation), while the event of an inventor or artisan's dedicating a work sample at a shrine is not without precedents in this period. But von Wilamowitz had no such approval to give to the accompanying letter. First, he argued, this presents an extended account of the device (iii) which would surely be superfluous, given the presence of the dedicatory model itself. Second, on points of detail it conflicts with the "genuine" part of the text (v), he alleged, as well as with the fragment from the *Platonikus* available to us through Theon and Plutarch; on the whole, however, it appears to assert little beyond what one might have read in the inscription or inferred from that. Third, this text was not known to Pappus, who wrote about two centuries before Eutocius, for his own version of Eratosthenes' method differs on points of detail in the construction and proof from the one given in the letter, although it presents a highly condensed version of materials parallel to Eutocius' items (iv) and (iii) or (v-a).¹³ Finally, and most damaging of all, the letter is banal: it is written without any sense of style and without any recognition of the occasion of its communication. One can hardly conceive, argued von Wilamowitz, that a man of such literary talents as Eratosthenes, writing to his patron, should frame his account in such a pedestrian manner. Thus, he concluded, the letter must be taken as a late forgery, still unknown at the time of Pappus; its author presumably transcribed from the votive monument the epigram of Eratosthenes and the partially mutilated description of the device and then, misled by the epigram's devotional invocation of Ptolemy as deity of the shrine, mistook the offering as a personal gift and so produced the letter as a companion explanatory document. But, he added, the ineptness of the forger's effort is betrayed on many points, most strikingly in its omission of second person forms of address almost until the very end. On the other hand, the composition has preserved for us the genuine epigram, for which we may be grateful.

Shall one accept this view of the provenance of the letter? To be sure, its style is concise, perhaps a bit abrupt; but it is not an illiterate production. One has yet to display such outright anachronisms which would mark it as impossible for a writer of the 3rd century B.C. Moreover, the historical and technical information it presents is entirely compatible with what we learn from our other sources on the early studies. Indeed, the absence of anachronisms relating to the mathematical content is impressive and would suggest at the very least a degree of thoughtfulness and skill on the part of the alleged forger. A comparison with the "genuine" parts of the document shows that the "forger" has not limited

himself to merely what has appeared in the epigram, but goes beyond it on several points in ways which indicate a real familiarity with this material, rather than mere fabrication. For instance, he observes that of the earlier methods only that of Menaechmus admitted any practical implementation "to a certain extent and that with difficulty" (*dyscherōs*); by contrast, the epigram refers only to the constructions by Archytas as "unwieldy" (*dysmēchana*), where in context the charge applies indifferently to the methods of both Menaechmus and Eudoxus as well.¹⁴

Relative to the argument concerning Pappus, no one has ever presumed that Pappus passed on *all* the sources at his disposal. Even if he did not possess a copy of the source in the form used by Eutocius, that hardly implies the nonexistence of that source at Pappus' time. After all, Eutocius himself presumably passed over the version in Pappus (if he had access to that) in favor of the text he actually reproduces. Surely, Pappus was capable of the same kind of editorial selectivity. As it happens, his text on Eratosthenes is rather more concise than that given by Eutocius, but on the whole not superior to it. Von Wilamowitz makes much of certain discrepancies (e.g., the shape of the sliding plates as triangular according to Pappus, but rectangular according to Eutocius)¹⁵ which have no bearing on the feasibility of the design and its realization. But he is silent on certain insights of Eutocius omitted by Pappus: e.g., that the "mesolabe" can be used for finding not just two, but any number of mean proportionals by the insertion of additional plates.¹⁶ The notion that Eutocius' report could result merely by transcribing and adapting what could be seen on the monument is simply not true, although it might well hold for the version in Pappus. In particular, Eutocius is far more detailed in his recommendations on the physical construction of the device. It is difficult enough to accept that the inscription should remain more or less intact as late as Pappus' time, early in the 4th century A.D.¹⁷; but that such a memorial of pagan worship could survive the volatile spirit of the later part of that century and the next is truly incredible.¹⁸ Von Wilamowitz' argument, that most of the information in the letter would be superfluous in the presence of the monument itself, is not compelling. One may consider the example of the ornate astrolabe invented and built by Synesius, disciple of Hypatia, statesman, philosopher and bishop: the specimen was forwarded to his noble associate Paeonius accompanied by a letter describing in detail the conception and physical execution of the device. Many of these details would be evident from the device, including the two epigrams engraved on it, which Synesius quotes in full: "let it be set down for such as may read it later, since for you it is enough that it lies on the tablet."¹⁹ One thus sees in Synesius' dedication a close parallel to Eratosthenes' letter on the "mesolabe."

The strongest indication of the letter's authenticity, however, would appear to be precisely what von Wilamowitz took to be its most questionable feature: its banality. For surely a forger would have let his imagination run wider than the narrow compass of factual materials given here. By contrast, those facts do amount to the kind of information which technical accounts accompanying other mathematical writings of this period contain. The prefaces to the treatises by

Archimedes, for instance, rarely go beyond stating the principal theorems in the work and making brief observations on the methods used.¹⁹ One would suppose that his correspondents Dositheus and Eratosthenes were receiving these communications by virtue of their official positions at Alexandria; but even this inference is not directly affirmed by anything actually said in the prefaces.²⁰ The *Sand-Reckoner* provides a good parallel to Eratosthenes' letter. It is devoted primarily to the exposition of the geometry and number theory needed for a specific computation dealing with the dimensions of the cosmos.²¹ Within it are details of the design of a sighting device by Archimedes himself and a report of particulars, both geometric and observational, relating to its actual use. The discussion elaborates a theme introduced through a general literary reference (i.e., the inadequacy of the account of the infinite by certain unnamed philosophers). In all these respects, then, it follows a pattern like that of Eratosthenes' letter. Indeed, both are framed as communications to royal patrons, yet the *Sand-Reckoner*, despite its much greater length, is as sparing in its use of second-person forms as the letter is.²² The Gelon, addressed as "king" by Archimedes, is believed to have served as regent in Syracuse for some thirty years, although he never reigned as king in his own right.²³ Was he still a youth or already a mature man when Archimedes wrote to him? Was the writing sent to him at Syracuse, or elsewhere? And where was Archimedes himself at that time: at home or abroad? Did he read the letter orally before an audience or was the writing intended for Gelon's personal study? Was Archimedes a kinsman of the regent or his former tutor? The document provides not a single clue toward the answer to any of these questions of context.

In view of this, how can one fault Eutocius' document for its failure to establish a setting? Von Wilamowitz' critique thus falters through his own failure to reckon with the character of the appropriate literary genre. Without grounds for suspicion, then, we may treat Eutocius' text no less seriously than any other, as a historical source. Indeed, von Wilamowitz' plea for admitting the epigram, falling within a genre of which he is a recognized master, would naturally prepare us for admitting the whole document as genuine. The burden of proof rests upon those who would maintain the contrary.

Under the view that the letter is authentic, its ostensible conflict with the alternative account from the *Platonicus* takes on entirely new significance. With reference to the efforts of the geometers in Plato's Academy, the letter says this:

[a] After some time they say that certain Delians fell into the same difficulty as they set about to double one of the altars in accordance with an oracle, and so sending out word they asked the geometers with Plato in the Academy to find for them what was sought. These applied themselves diligently and sought (how) to take two means of two given lines, [b] and Archytas of Taras is said to have discovered (a solution) by means of semicylinders, Eudoxus by means of the so-called curved lines. [c] But it happened that these all wrote in demonstrative fashion, none being able to manage it in a practical way or to put it into use, save to a certain small extent Menaechnus and that with difficulty.²⁴

It would appear that two or three different sources have been stitched together:

(a) an account of the legend of the Delian oracle and the communication to the Academy; (b) a reference to solutions by Archytas and Eudoxus; and (c) an account somehow acquainted with writings derived from these geometers, in particular, from Menaechnus. Doubtless, communications between Archytas and the Academy were consistently good, through visits by Plato and others to Italy, and through correspondence; but the ancient biographical traditions do not mention an actual residence by Archytas at the Academy.²⁵ Thus, line (b) reads uncomfortably as a simple continuation of (a). Further, both (a) and (b) are reported at some detachment from primary sources ("they say;" "it is said"). By contrast, (c) seems to speak from a certain familiarity with such sources. When Eutocius elsewhere presents Archytas' method, Eudemus is named as source.²⁶ One would suppose that Eudemus also treated the methods of Eudoxus and Menaechnus and that his versions were available to Eratosthenes. That being so, the tentative tone of line (b) is puzzling: surely it would be *obvious* that Archytas used semicylinders and Eudoxus used curved lines. Here then is a place where an interpolator may have inserted a line, based on his reading of the allusions to these geometers in the epigram.²⁷ Without it, line (c) follows as a fully appropriate technical observation after the story in (a).

The other account of this incident is transmitted by Theon of Smyrna in the introductory section of his compilation of mathematical materials pertinent to the study of Plato:

For Eratosthenes says in his writing the *Platonicus* that when the god pronounced to the Delians in the matter of deliverance from a plague that they construct an altar double of the one that existed, much bewilderment fell upon the builders who sought how one was to make a solid double of a solid. Then there arrived men to inquire of this from Plato. But he said to them that not for want of a double altar did the god prophesy this to the Delians, but to accuse and reproach the Greeks for neglecting mathematics and making little of geometry.²⁸

This version of the story is further attested in two passages from Plutarch, for whom the moral is that one needs dialectical skill to maneuver among the ambiguities of oracles.²⁹ Following von Wilamowitz, many have taken these discrepancies to militate against the authenticity of the version in Eutocius. One is thus to suppose that the forger of the letter diluted Eratosthenes' account in the preparation of his own. But why should he have sought to remove all those details which give the story vividness: the occasion of the plague and Plato's personal intervention to interpret the true meaning of the oracle? The story in the letter seems to have no point beyond explaining why a certain group of geometers came to concern themselves with cube duplication.

The embellishments reported by Theon and Plutarch suggest that their versions derived from the elaboration of the more pedestrian account cited by Eutocius. Doubtless, Eratosthenes himself was responsible for these changes, as he sought in his *Platonicus* to dramatize aspects of Plato's view of mathematics. After all, motives of this very sort led Theon and Plutarch to cite this passage in their own works. Elsewhere, Plutarch exploits the cube duplication to illustrate another

aspect of Plato's philosophy: that geometry is not a matter of perceptible things, but only of the eternal and incorporeal. Thus, he has Plato criticize the mechanical solutions of Archytas, Eudoxus, and Menaechmus as spoiling the true good of geometry.³⁰ This alternative version echoes the second part of Eutocius' account, where the practicability of these geometers' solutions is brought up for comment. But there is a patent difference: according to Eutocius, Eratosthenes criticizes the earlier efforts for their being too abstract; by contrast, Plutarch derives from him a criticism of their overly mechanical character. Of course, this discrepancy need only be one of emphasis. The surviving accounts of these methods are in fact fully geometrical in the formal manner; but they do rely on certain mechanical conceptions, like the generation of curves by moving lines or by the intersection of solids of revolution.³¹ The student of Plato's abstract philosophy might well view the latter with disapproval. On the other hand, Eratosthenes' own solution entailed the production of a mechanical device, and in his eyes its practical viability is a major asset; in this context, the practical limitations of the earlier designs could well be a point of note, inducing him to underscore their purely theoretical nature. Here again, the discrepancy is undoubtedly due to Eratosthenes himself. The intent in the letter is historical, but that in the *Platonicus* is dramatic and philosophical. He might thus take certain liberties in the latter, denied to him in the former. To set the scene for his dialectical point, he can elaborate the mechanical elements more explicitly than the original treatments actually did. Such fictionalization would surely be acceptable in a work which made no pretension of being plainly historical.

Accepted as a serious historical source by a man well placed for a knowledge of this period of Greek mathematical history, Eratosthenes' text as reported by Eutocius offers hope of insight into matters often obscured in modern discussions. First, although historians typically wish to leave open the question of the historical validity of the Delian oracle,³² one now readily perceives the story to be a fabrication, originating from within the Platonic Academy around the middle of the 4th century B.C. By this time the problem of cube duplication was already familiar, for as Eratosthenes himself notes, important advances in its analysis were made by Hippocrates of Chios, that is, almost a half-century before.³³ It would seem odd indeed for the Delian oracle to be concerned with an old problem which then happened to be eluding the efforts of contemporary geometers. But on the other hand, as a dramatic way of affording recognition and motivation to those efforts, the story makes good sense, especially since geometers associated with the Academy were prominent in this activity. But one can only guess what the context and purpose of the story might have been at its first composition. As we have indicated, the specific morals attached to it by Eratosthenes (in the *Platonicus*) and by Plutarch seem to be their own additions, but they are likely to have sensed accurately the ancient Academicians' view of these matters. Any student of the *Republic* can appreciate the strength of Plato's insistence on purity and abstractness of mathematical entities.³⁴ The kinematic element in the constructions for the cube duplication might thus provoke discussion of the relative

status of geometry and mechanics, and indeed this very issue is important within Aristotle's theory of the order of the sciences.³⁵

Furthermore, Eutocius' text of Eratosthenes is noteworthy for the sparseness of the historical information it transmits. One should have thought that Eratosthenes would be concerned with producing as full an account as he could of the studies preceding his own, and his text does leave that impression. Yet it is amazing that he turned up with so little. He appears to have discovered nothing worth mentioning from the whole century or so separating him from the efforts of the Academicians. As for the even earlier efforts, he cites only the work of Hippocrates and a scene from an unnamed tragic poet.³⁶ In the latter, when King Minos is told that the tomb he had ordered for Glaucus will have dimensions of one hundred feet on a side, the king answers that this would be too small, and so, "let it be double; without mistaking its fine form, swiftly double each member of the tomb." The new structure will of course be eight times larger than the original one in volume, not its double. Eratosthenes calls attention to the poet's error. But actually there is none here: the poet intends that each dimension shall be doubled; he is not articulating the mathematicians' problem of the cube duplication. Evidently, Eratosthenes has misconstrued the passage in his desire to find precedents for an interest in this problem.

In effect, Eratosthenes has sought some motivation for Hippocrates' study of the cube duplication. But Hippocrates' effort by its very nature reveals that motive; for in Eratosthenes' account, this is what he does:

It used to be sought by geometers how to double the given solid while maintaining its shape. . . . After they had all puzzled for a long time, Hippocrates of Chios was first to come up with the idea that if one could take two mean proportionals in continued proportion between two lines, of which the greater is double the smaller, then the cube will be doubled. Thus he turned one puzzle into another one, no less of a puzzle."

Hippocrates' insight is of course not restricted to lines assumed in a 2 : 1 ratio. If for any two given lines, A and B, we can insert the two mean proportionals, X and Y, then $A : X = X : Y = Y : B$. Thus, by compounding the ratios, one has $(A : X)^3 = (A : X)(X : Y)(Y : B)$, that is, $A^3 : X^3 = A : B$. Thus, X will be the side of a cube in the given ratio (B : A) to the given cube (A^3).³⁷

In his closing line here, Eratosthenes seems to perceive little merit in Hippocrates' move. But in fact, by this stroke Hippocrates has put the problem into a form permitting the application of a whole new range of geometric techniques, those of proportion theory. In general, this procedure of "reducing" (*apagōgē*) one problem to another from whose solution its own would follow is a powerful technique of problem solving, a forerunner of the fruitful method of geometric "analysis." As Proclus tells us, Hippocrates was the first geometer known to have applied the method of reduction in the investigation of "puzzling diagrams," i.e. difficult problems of construction.³⁸ We will see later that Hip-

hippocrates appears to have adopted a similar strategy in the investigation of the circle quadrature.

Later geometers clearly recognized that such a *reduction* does not in itself amount to a *solution* of the problem posed. But would Hippocrates already have made this distinction in the case of his treatment of the cube duplication? A passage from Aristotle suggests that he did. In *de Anima* II, Ch. 2, Aristotle observes that one may define a term like "squaring" (*tetragōnismos*) not only by saying *what* it is (i.e., the equality of a square with a rectangle), but also by stating its *cause* (i.e., the finding of a mean proportional).⁴⁰ Now, the latter refers to the reduction of the given problem of areas, such as one may read in Euclid's VI, 17, and is the precise plane analogue to Hippocrates' problem of solids. But the construction of this planar problem, whether in the form of a squaring (as in *Elements* II, 14) or in that of finding a mean (as in VI, 13), was certainly already available to Hippocrates. A long fragment of his study of the quadrature of crescent-shaped figures survives, and the techniques of proportions are applied throughout to situations far more demanding than the squaring of rectangles.⁴¹ Since the distinction between the reduction and the solution was clear in the instance of the planar case, the same must surely have been appreciated in the solid case as well. We may suppose, then, that Hippocrates and his followers continued their research into the cube duplication by seeking the construction of the two mean proportionals, where the latter effort achieved its first clear successes through the discoveries of Archytas, Eudoxus, and Menaechmus.

These considerations help clarify the way in which the cube duplication was first articulated as a problem. Within the study of the measures of plane figures, affected via the techniques of proportions, it would be perfectly natural for a geometer like Hippocrates to consider the case of two mean proportionals and so perceive its relation to the problem of volumes, by analogy with the case of the single mean. In this way, the new problem emerges through the natural development of geometric research. Now, it sometimes happens that research moves in new directions in response to external stimuli: practical problems arising from commerce, government, and engineering are a manifest basis of much in the most ancient mathematical traditions; ritual might sometimes play a role; or philosophy might instill a sensitivity to formal questions.⁴² The story of the Delian oracle has thus often been used as an example of such external motivation. But one can now see that the ancient evidence on the origins of the cube duplication affirms quite a different view: that the legend arose within Plato's Academy around the middle of the 4th century B.C., long after the problem had attained notoriety through the work of Hippocrates; and that Hippocrates himself articulated this problem as a natural extension of the geometric field of his time. In the absence of direct testimony to earlier work in the 5th century, one has no grounds for supposing a development of any other sort.

THE QUADRATURE OF THE CIRCLE

In connection with the problem of constructing in a given angle a parallelogram equal to a given rectilinear figure (*Elements* I, 45), Proclus remarks:

Having taken their lead from this problem, I believe, the ancients also sought the quadrature of the circle. For if a parallelogram is found equal to any rectilinear figure, it is worthy of investigation whether one can prove that rectilinear figures are equal to figures bound by circular arcs.⁴³

Although the words "I believe" reveal this view to be merely a surmise on Proclus' part, rather than a conclusion based on more or less explicit testimony, it is far from unreasonable. Indeed, there appears to be a certain sense of the problem of circle quadrature in remnants of the ancient Egyptian and Mesopotamian traditions as far back as the middle 2nd millennium B.C. and earlier. Specifically, the scribe of the Rhind Papyrus works out the area of a given circle first by subtracting $1/9$ of its diameter and then squaring the result.⁴⁴ Thus, in effect, the circle is equated with the square whose side equals $8/9$ the diameter of the circle. This rule is illustrated by a figure in which a square is divided into nine equal squares, where the diagonals of the four small squares at the corners are drawn in. In this way one sees at once that the circle might be approximated as $7/9$ the enclosing square. One might conjecture that the scribe went on to subdivide each of the squares into nine squares and then tried to estimate how many of the 81 resulting squares combined to approximate the circle. But another approach might have been followed. Knowing that problems of areas often require computing squares and their roots, the scribe might have sought a value near $7/9$ which resulted from a squaring. Since $7/9 = 63/81$, where the latter is just short of $64/81 = (8/9)^2$, a quasi-deductive route might be possible for deriving the rule applied by the scribe of the papyrus.

If such was indeed the basis of the rule, the scribe would also have a basis for recognizing the measurement of the circle as a geometric problem. The same would not be true within a tradition relying entirely upon empirical derivations of its geometric rules. It is hardly possible, for instance, that the Egyptians could have known the $1:3$ ratio of the volumes of the cone and cylinder having equal altitude and base other than by observing that the fluid contents of a conical vessel can be emptied exactly three times into a cylindrical one, or by some comparable physical measure.⁴⁵ But inaccuracies of manufacture, spillage, and other such factors would render this procedure approximate. There would thus be no firm demarcation between rules which are exact (as is that for the ratio of volumes) and those which are approximate (e.g., the circle measurement). Certainly, within the early traditions the aim was to obtain practical procedures. The project of somehow producing a construction of the square equal to the circle seems remote from this objective, even if it may be perceived as entailed

within the results given. Its explicit formulation as a problem must then be sought elsewhere.

We may gather that Greek mathematicians had hit upon this formulation of the problem before the close of the 5th century B.C. For in his comedy *The Birds* (staged in 414 B.C.) Aristophanes presents this lampoon of the astronomer Meton arriving as self-appointed civic surveyor of Cloud-cuckoo-land:

If I lay out this curved ruler from above and insert a compass—do you see?—
... by laying out I shall measure with a straight ruler, so that the circle becomes square for you."⁴⁰

Now, there is no report of Meton's inquiring into the circle quadrature, and the passage here actually views him as somehow dividing the circle into quadrants.⁴¹ But the force of the jest would seem to require that the audience perceive it to refer to a real problem under investigation by geometers at that time. This conforms with reports we have of certain efforts by Hippocrates, Antiphon, and Anaxagoras.

Relative to Hippocrates, Aristotle tells that he proposed a false proof of the quadrature of the circle "by means of segments" or "by means of lunules."⁴² The latter refers to Hippocrates' masterful construction and quadrature of crescent-shaped figures bounded by arcs of circles; we turn later to the question of the bearing this effort had on the circle quadrature. According to the account preserved by Simplicius, the 6th-century Aristotelian commentator, drawing explicitly from a report by Eudemus, Hippocrates prefaced his treatment with demonstrations of certain basic theorems on circles:

He made his start and set down first some things useful for those [constructions], that segments of circles have to each other the same ratio as their bases in power; and this he proved from having proved that the diameters have the same ratio in power as the circles....⁴³

This passage notes explicitly that Hippocrates *proved* these theorems. Moreover, it goes on to supply further details of the derivation of the segment theorem in good agreement with a version given much later by Pappus.⁴⁴ We may thus be confident that toward the close of the 5th century, Hippocrates already appreciated the essential concerns of the deductive form entailed by the proofs of theorems such as these. Now, the lines just cited include the statement of a theorem identical to one in Euclid's *Elements* (XII, 2): "Circles are to each other as the squares on their diameters." Simplicius' terminology of "powers" (*dynameis*) instead of "squares" (*tetragōna*) reassures us that here he has followed the archaic Eudemean usage in his text, rather than improvise through his own knowledge of Euclid.⁴⁵ But we encounter difficulty in supposing that Hippocrates advanced the proof of this theorem on circles in its Euclidean form. For the limiting method on which that depends is due to Eudoxus, as we learn on the good authority of Archimedes.⁴⁶ Then what sort of proof could Hippocrates have provided a half-century or more before Eudoxus?

Aristotle's remarks on the false efforts at circle quadrature link Hippocrates

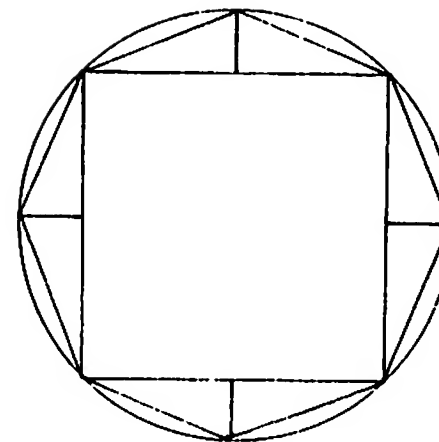


Figure 1

with two sophists from early in the 4th century, Bryson and Antiphon.⁴⁷ What Simplicius reports of the latter may help us here. Antiphon proposed to construct a rectilinear figure equal to a given circle by a procedure involving the successive doubling of the number of sides of an initial inscribed polygon. For instance, if one starts with the inscribed square, erects the perpendicular bisector to each side, marks the point where each meets the circle, and connects each of these points to the adjacent vertices of the square, one will have erected an isosceles triangle on each side of the square, and their combination with it will produce the inscribed regular octagon. (See Figure 1.) One may next bisect the sides of this octagon, ultimately resulting in the inscribed 16-gon; and so on.

Continuing in this way, at the point where the area was exhausted, he said he will have inscribed a certain polygon in the circle in this manner, such that its sides conform to the arc of the circle on account of their smallness. Since we are able to produce the square equal to any polygon... because the polygon has been produced conforming to the circle, we shall also have produced a square equal to the circle.⁴⁸

Antiphon's step yielding an arc conforming (*epharμοζειν*) to the sides of the inscribed polygon would appear to be drawn from notions familiar through the ancient atomists. For them, the microscopic world of atomic reality might be qualitatively different from the macroscopic world as perceived by our senses. Specifically, the circle appears to meet its tangent at a single point only, and we insist on this as part of the abstract conception of this configuration; but in their physical manifestation they will meet along a small line segment. This very question was discussed by Democritus and some of the sophists, notably Protagoras; and one of Simplicius' sources on Antiphon, the 3rd-century-A.D. commentator Alexander of Aphrodisias, observes that this principle—the circle's meeting the tangent line at one point only—was the one whose abolition lay at

the root of Antiphon's fallacious quadrature.⁵³ We thus perceive a firm dialectical context for Antiphon's argument.

The invalidity of his circle quadrature is evident, for it relies upon the actual "exhaustion" (*dapanan*) of the whole area lying between the circle and the polygons, while the indicated procedure of inscribing polygons will not do this if applied only a finite number of times.⁵⁴ Hence, for Aristotle and the commentators, its refutation is not a matter for the geometer to be concerned about. By contrast, modern scholars have tended to assign to Antiphon credit for the introduction of the important notion of polygonal approximation, essential for the ancient quadratures of curvilinear figures.⁵⁵ But is this a plausible position to maintain? Given that Antiphon and Hippocrates were near contemporaries, the latter without doubt being rather the older, shall we suppose that the sophist, in the course of framing a muddled geometric argument to support his dialectical position, lent essential insights to the geometer? Or is it not more likely that he was drawing from and modifying procedures already familiar among geometers? The latter certainly typifies the later philosophical tradition, as represented by Plato, Aristotle, the Stoics, and the Sceptics, for instance, and their stance toward the technical sciences of their times. By contrast, it seems unclear why a geometer like Hippocrates should turn to the sophists for technical assistance in a theorem of this sort. The fact that Antiphon's use of this technical procedure lapses into patent fallacy, while Hippocrates requires the same procedure for the proof of a theorem implemented with full logical precision in his quadratures of lunules, merely confirms one's doubts about Antiphon's originality in this technical context.

If we assign to Hippocrates knowledge of the polygonal procedure preserved in our text of Antiphon's quadrature, then we obtain a good basis for reconstructing a proof he could have used for the theorem on the ratios of circles. For one notes a subtle difference between Antiphon's method and the one used in the Euclidean proof of this theorem (XII, 2). Euclid obtains each polygon in the sequence by bisecting each of the arcs cut off by the vertices of the polygon preceding it. Antiphon, on the other hand, builds each new polygon from its precursor by adding the isosceles triangles onto each of its sides. Now, if one considers two circles, each enclosing similar polygons constructed in this way, since the initial polygons and the added triangles taken pairwise are similar, these figures will be in the same ratio, namely that of the squares of the diameters of the circles. Hence, the *sum* of *all* these figures, extended indefinitely, will be in the same ratio. Since these sums constitute the circles themselves, it follows that they too have the ratio of the squares of their diameters, as Hippocrates asserts.⁵⁶

From the strict logical viewpoint, of course, this argument fails, for it applies to the infinite case a theorem on the ratios of sums which can be established for sums of only a finite number of terms, if even that, given the technique of proportions available to Hippocrates.⁵⁷ But this approach is so strongly founded on clear notions of the continuity of magnitude that the difficulty might not necessarily have been appreciated at an early stage of these studies. We must assign to Eudoxus the first full awareness of this difficulty and the discovery of

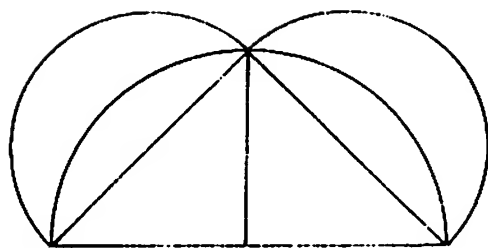
the method of limits by which it could be removed. It is noteworthy that the technique of sums we have just proposed is a standby within Archimedes' demonstrations; that Archimedes provides a proof, but only for the finite case (*Conoids and Spheroids*, Prop. 1); nevertheless, that he applies its infinite case in all of the planar and solid measurements presented in his *Method*.⁵⁸

One cannot say for certain whether such studies of the circle quadrature can be set much before the time of Hippocrates. Writing in the 2nd century A.D., Plutarch mentions that the 5th-century natural philosopher Anaxagoras "drew the quadrature of the circle" during a spell in prison.⁵⁹ This would place an awareness of the theoretical character of the problem of circle quadrature to not much later than the middle of the 5th century B.C. It is an especially intriguing attribution, in that Anaxagoras insisted on the continuity of matter as the basis of the nature of things and seemed to perceive some of the mathematical implications of the principle of indefinite divisibility, the very principle later violated in Antiphon's circle quadrature. To be sure, Proclus reports that Anaxagoras "touched on many things relating to geometry."⁶⁰ But the ancient tradition otherwise assigns to him no specific interest in mathematics, and Plutarch's testimony is quite casual, inserted merely to illustrate the point that the thinking man finds happiness even under trying circumstances. The subsequent notoriety of the circle quadrature might all too easily color the testimony transmitted by later authorities. Thus, we must forswear using Anaxagoras' alleged contribution as a sign of early interest in the problem.

From Aristotle we may gather that a construction of the circle quadrature was attempted on the basis of Hippocrates' quadrature of lunules. For in his account of the logical method of "reduction" (*apagōgē*) he remarks thus:

We are the closer to knowing, the fewer the mean terms in the syllogism. For instance, as the circle together with the lunules is equal to a rectilinear figure, we are that much the closer to the squaring of the circle itself.⁶¹

Viewed in the light of Hippocrates' reduction of the cube duplication to the problem of the two mean proportionals, this passage seems to indicate that Hippocrates attempted a similar strategy for the circle quadrature: to cast it into an alternative form permitting the application of a wider range of the techniques then known. This is fully borne out in the extended discussion of circle quadrature presented by Simplicius commenting on a related passage from the *Physics* where Aristotle mentions the false quadratures.⁶² In addition to sketching and criticizing Antiphon's argument on the circle quadrature, as we have already seen, and noting several other efforts on this problem, Simplicius gives a detailed account of Hippocrates' quadrature of lunules. Indeed, he provides two different versions, a simplified treatment known through Alexander's commentaries, followed by an elaborate series of constructions and proofs quoted verbatim from Eudemus' history. This passage from Eudemus is our most important fragment from the pre-Euclidean geometry, providing invaluable insight into the content, methods, and terminology of the early tradition. We shall examine it in detail, after a brief consideration of the shorter version from Alexander.

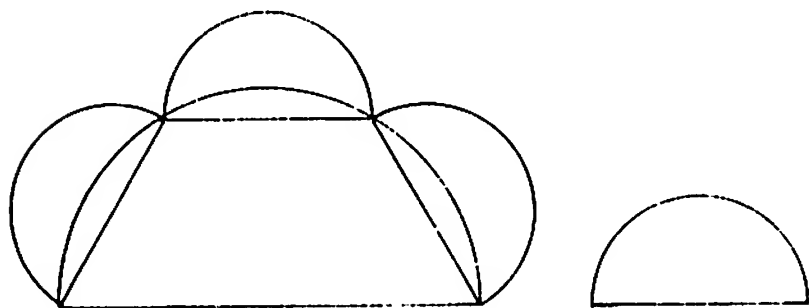


(a)

Figure 2

In Alexander's text, two cases of lunules are presented⁶⁷: (a) Starting from a right isosceles triangle, we draw on each of its sides a semicircle. (See Fig. 2a.) Since the square on the hypotenuse equals the sum of the squares on the two legs, and since circles are as the squares on their diameters, it follows that the semicircle on the hypotenuse equals the two semicircles on the legs. If we now remove the portion common to the large semicircle and the two smaller ones, we find that the isosceles triangle which remains is equal to the two lunules which arch over its legs. Thus, by equating the lunules to a rectilinear figure, we have effected their quadrature.⁶⁸ (b) In the second case, we begin with the trapezium formed as a bisected regular hexagon and draw semicircles on each of its sides. Since the longest side of the trapezium is double each of its shorter sides, the large semicircle equals four of the smaller ones. As before, subtracting the common portion, we find that "the remaining lunules together with the (small) semicircle are equal to the trapezium." (See Fig. 2b.)

Recalling Aristotle's remark on the reduction of the quadrature of the circle to that of the lunules, cited above, we see that he evidently had this construction of Hippocrates in mind. But precisely what conclusion Hippocrates himself intended to draw from it has been debated among ancient and modern con-



(b)

Figure 2

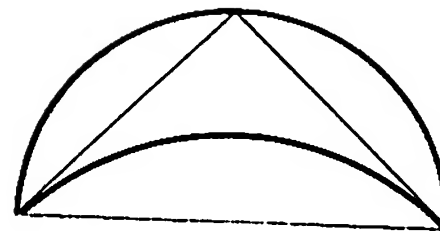


Figure 3

mentators alike.⁶⁹ Alexander, for instance, supposed that Hippocrates wished to argue that since the lunule in (a) has been squared, while the sum of the circle and the lunule in (b) has also been squared, then the circle itself has been squared. If so, he would have committed the patent error of assuming that the quadrature of a single case of lunule amounts to the quadrature of all cases. Simplicius, by contrast, found it incredible to conceive that a geometer of Hippocrates' outstanding caliber could have lapsed into such a blunder. Under this view, one would charge to others, for instance, to Aristotle, the attempt to construe Hippocrates' results as a circle quadrature. Doubtless, this is the correct view, for as we saw in the parallel instance of the cube duplication, Hippocrates' reduction of the problem there was not likely to have been regarded as a solution of it. As we shall see later, Simplicius' text provides further insights into Hippocrates' view of the circle quadrature.

The alternative account, which Simplicius quotes "word for word" (*kata lexin*) from Eudemus, presents the full treatment of four cases of lunules. The first is essentially identical to Alexander's case (a), while the fourth bears comparison, if somewhat more loosely, with his case (b). The other two cases are new, constructing lunules which are in the one instance greater and in the other less than the arc of a semicircle. Translations and technical summaries of this text are readily available, so that a brief summary will suffice here.⁷⁰ My principal aim now is to retrieve the line of thought underlying Hippocrates' constructions and thus to reveal his sense of the more general class of lunules.

(i) In the first case, Hippocrates begins with the right isosceles triangle and circumscribes the semicircle on its hypotenuse. (See Fig. 3.) This much conforms with case (a). But now, instead of drawing semicircles on the legs also, he draws on the hypotenuse the circular segment similar to those which appear on the legs of the triangle. In this way he obtains a single lunule, the figure bounded by the two circular arcs, and shows that it equals the initial triangle. This follows, since the similar segments are as the squares on the chords which define them, so that the larger segment equals the sum of the two smaller ones. If to the mixtilinear figure inside we add the two small segments, we obtain the lunule; if we add to it, alternatively, the large segment, we obtain the triangle. Thus, the triangle and the lunule are equal to each other. It is clear that this lunule is the same as each of the two lunules constructed in case (a). But the present treatment reveals a greater technical range. For instance, its author operates freely with similar

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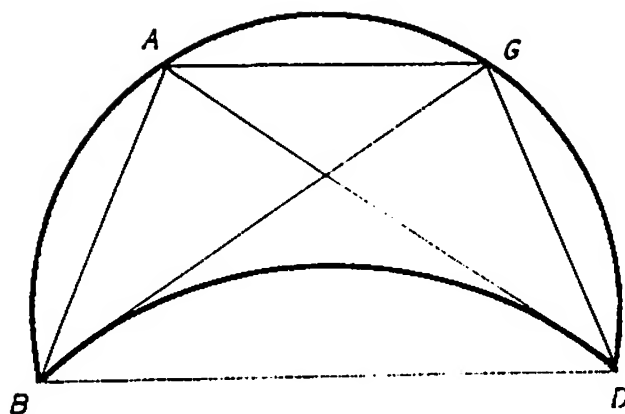
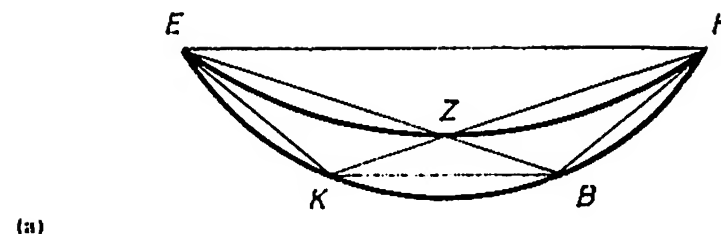


Figure 4

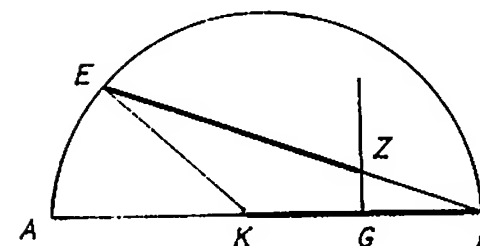
segments, not just semicircles. The pertinent definition, given earlier in the text, stipulates them as subtending equal angles in their respective circles. As we saw above, his proof that similar segments, like similar rectilinear figures, have the ratio of the squares on homologous sides signifies that major progress has already been made toward the Eudoxean manner of measuring curvilinear figures. In all these respects, the treatment of case (i), despite its relative simplicity, prepares us nicely for the more intricate constructions which follow.

(ii) Hippocrates now assumes the construction of a trapezium whose base has to each of its three remaining equal sides the ratio $\sqrt{3} : 1$. (See Fig. 4.) He next draws the circumscribing circle and then draws on the base the circular segment similar to those which have thus been formed on the three other sides. The text establishes in full detail that the arc of the outer bounding circle is greater than a semicircle.⁶⁹ The actual proof of the quadrature of the lunule is omitted from Eudemus' text, so Simplicius supplies his own, remarking that Eudemus must have considered it obvious. But in fact the analogous proofs are given in full for each of the following two cases, and textual considerations show these to be in the pre-Euclidean manner of Eudemus.⁷⁰ Thus, one may infer that Simplicius' text came to lose portions of the proof through scribal omissions.⁷¹ At any rate, since the one large segment equals the three small ones, the same process of addition as used in (i) shows that the lunule equals the trapezium.

From these two cases we may recognize the condition Hippocrates has in mind for the construction of squarable crescents. If we conceive of a rectilinear figure bounded by two polygonal arcs, the outer consisting of n sides each equal to s_n , the inner consisting of m sides each equal to s_m , and if $s_n^2 : s_m^2 = m : n$, and if furthermore the sides s_n, s_m subtend equal angles in the circles circumscribed about their respective polygonal arcs, then the lunule bounded by these circular arcs will equal the rectilinear figure bounded by the polygonal arcs. Although Hippocrates does not formulate the construction in such general terms,



(a)



(b)

Figure 5

his treatment of the third case indicates that a conception of this sort did indeed guide his thought. His examination of this next case, like those which preceded it, is entirely in the synthetic mode, and so tends to obscure the manner of its discovery. I shall thus attempt to reconstruct on its basis an alternative version in the analytic mode in order to display its essential idea.⁷²

Let an investigation of the lunule of the type just stated be proposed, where the two bounding arcs correspond to polygonal arcs, the one of three equal sides, the other of two. Thus, $s_2 : s_1 = \sqrt{3} : \sqrt{2}$. Let us conceive that the figure has been produced, consisting of the double arc EZH and the triple arc EKBH (see Fig. 5a); if we pass a circle through each of these polygonal arcs, then we must also have that the circular arcs EZ, ZH, EK, KB, BH all subtend the same angle at the center of their respective circles. We join EH and consider the angles ZEH and KEH. As the former intercepts the single arc ZH, while the latter intercepts the double arc KBH, it follows that angle KEH is double ZEH, so that EZ bisects angle KEH and its extension must pass through B. Furthermore, point Z lies on the perpendicular bisector of BK⁷³, while point E lies on the circle of radius EK and center K. Thus, given lengths in the ratio $\sqrt{3} : \sqrt{2}$, to accomplish the construction we need only take the shorter length as KB, draw the circle having the same length as radius and center at K, draw the perpendicular bisector of KB, and then so place a line passing through B that the portion of it intercepted between the circle and the perpendicular bisector equals the larger of the given lengths (see Fig. 5b). In this way, both the placement and the length of EZ will be determined, and from it the rest of the figure can be completed.

The latter sets out precisely the same form in which Hippocrates accomplishes

the construction of case (iii) of the lunules. In particular, one perceives through this analysis how natural it is to use the placement of EZ for effecting this. In Hippocrates' terms, "the line *inclining (neuoussa)* toward B and intercepted between the line DG and the arc AEB shall have the given length."⁷⁴ This is the earliest known instance of the use of the constructing technique called *neusis* by the Greeks. As we shall later see, it plays a prominent role in many of the problem-solving efforts by Archimedes and his followers a century and a half after Hippocrates.⁷⁵

Having so constructed the lunule, Hippocrates next proves that it is equal to the associated rectilinear figure EZHBK; for since $3 EK^2 = 2 EZ^2$, the three outer segments will equal the two inner ones. Since, furthermore, "the *méniskos* (lunule) consists of the three segments together with that part of the rectilinear figure apart from the two segments,"⁷⁶ the lunule and the rectilinear figure are equal. Hippocrates goes on to show that the outer bounding arc of this lunule is less than a semicircle. At this point Eudemos injects an amazing remark:

In this manner Hippocrates squared every lunule, since in fact (*eiper*) [he squared] the [lunule] of a semicircle and the one having the outer arc greater than a semicircle and the one [having the outer arc] less than a semicircle. . . . But he squared a lunule together with a circle as follows."

What are we to make of "every lunule"? Does it signify merely the three particular cases just examined? But in context a more general sense seems to be indicated, since one moves at once to the consideration of the case pertaining to the circle quadrature. Then has Eudemos erred in supposing that squaring one lunule or even a few is tantamount to squaring them all? If this is so, then did Hippocrates himself make the same error, thus bringing on himself Aristotle's charges of having produced a fallacious argument on the circle quadrature? But it is hard to imagine how the one responsible for these carefully ordered proofs on the lunules could then have committed such an elementary blunder in reasoning. Now, there is indeed a sense in which Hippocrates might claim to have established the quadratures of a whole class of lunules, namely those for which the *m* similar segments along one bounding arc equal the *n* similar segments along the other arc. Although, of course, he has not produced the general *construction* of these figures, he can assume that their *quadratures* are obvious by virtue of the three cases given. If this is what Hippocrates intended by asserting that "every lunule" had been squared, we would infer that Eudemos, and doubtless others before him, had misconstrued his claim as referring to every possible form of lunule.

As for case (iii), so also for (iv), the synthetic mode adopted by Hippocrates conceals its essential line of thought. But it emerges if one views this case in relation to case (b) in Alexander's version, by analogy with the relation of cases (i) and (a). If we start with the hexagon inscribed in a circle, instead of using either its side or its diameter as base, let us introduce the chord connecting the two nonadjacent vertices H and I. (See Fig. 6.) Thus, HI is the side of the

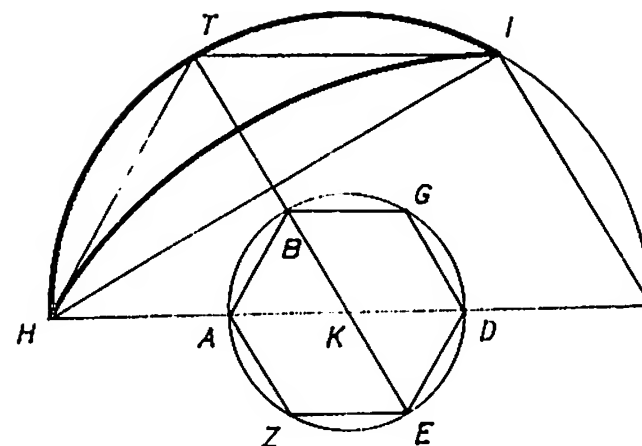


Figure 6

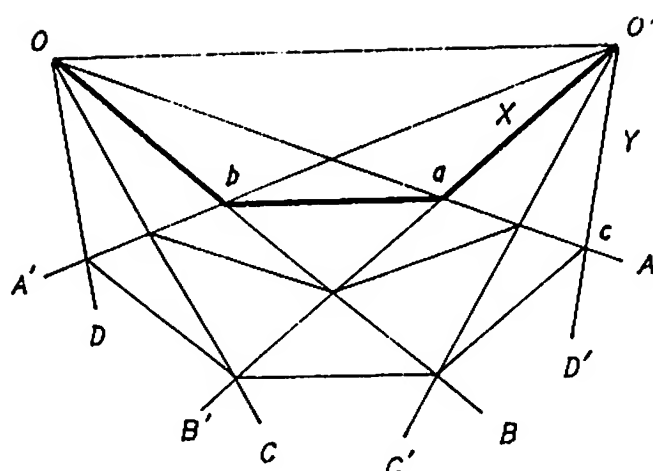
inscribed equilateral triangle, while HT is the side of the hexagon, so that $HI : HT = \sqrt{3} : 1$. If we draw over HI a segment similar to each of those over HT, TI, it will equal three times either of those segments. In the familiar manner, it follows that the triangle HTI equals the lunule together with one of those smaller segments. Thus, squaring the lunule reduces to squaring that segment. Hippocrates conceives of a set of segments similar to it and equal to it in sum; these are the segments on the sides of a hexagon inscribed in a circle whose diameter has to that of the initial circle the ratio $1 : \sqrt{6}$. The segment over HT thus equals the sum of all six segments on the sides of the hexagon ABGDEZ. Adding to both the area of the hexagon and then combining with the earlier result, we find that the triangle together with the hexagon is equal to the lunule plus the smaller circle. In Eudemos' text the construction begins with the drawing of the two circles, so that the motivation behind the choice of $1 : \sqrt{6}$ as the ratio of their diameters remains unclear until well into the proof. Having established the equality just stated above, he concludes:

Thus, if the rectilinear figures [i.e., the triangle and the hexagon] mentioned can be squared, so also can the circle with the lunule."

This ends the quotation from Eudemos; for Simplicius observes in the very next sentence:

One ought to place greater trust in Eudemos [than in Alexander] to know things pertaining to Hippocrates of Chios, for he was nearer to his times, being a disciple of Aristotle.

It is clear in Eudemos' account that Hippocrates makes no presumption of having squared the circle itself, but only the circle in combination with the lunule. This

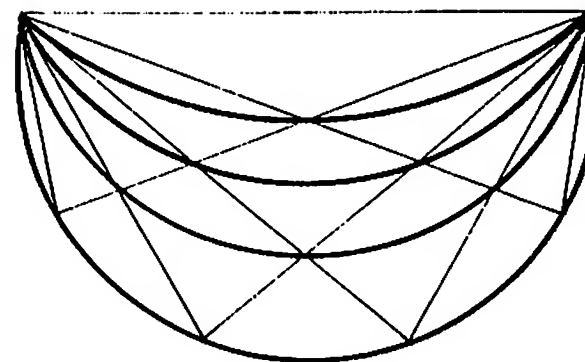


(a)

Figure 7

is so clear, in fact, that Simplicius questions the view that Hippocrates intended this here at all, and so suggests that when Aristotle speaks of his fallacious quadrature of the circle "via segments," he must refer to an entirely different argument.⁷⁹ Far more likely, however, is that Hippocrates presented this last case of lunule as a successful *reduction* of the circle quadrature, without its being a *resolution* of that problem. He would thus leave for later research the investigation of this lunule. If some followers rashly asserted that the circle had indeed been squared in this way, we have no grounds at all for charging this error to Hippocrates himself or even for suggesting that he wished slyly to leave this impression.

Aside from Hippocrates' attitude toward the circle quadrature, we should consider what his view of constructions was. It will be helpful to consider first the more general class of lunules and certain modern contributions to their study. At each end of the given line segment OO' let there be drawn a system of equally spaced rays, OA, OB, OC, OD, \dots and $O'A', O'B', O'C', O'D', \dots$, respectively, where in each case the angle θ separating consecutive rays is the same. (See Fig. 7a.) Then we may form a polygonal arc from O to O' by selecting the intersections of appropriate rays from each system. For instance, if point a is the intersection of rays $OA, O'B'$ and point b is that of $OB, O'A'$, then the broken line $O'abO$ will be a polygonal arc consisting of three equal lengths; for if we pass through them the circular arc joining O and O' , each length will be a chord subtending angles equal to θ relative to the points O and O' . Thus, the circular segments having these chords as base will be similar to each other. (See Fig. 7b.)⁸⁰ In analogous fashion, we can construct between O, O' another polygonal arc having as many equal elements as desired.⁸¹ As before, we obtain a set of circular segments similar to each other and to the segments associated with any other such polygonal arc constructed via the same initial sets of rays. Let two such arcs be taken, the inner having n elements, the outer having m elements. Denoting by X a side of the inner arc (e.g., $O'a$) and by Y a side of the outer



(b)

Figure 7

arc (e.g., $O'c$), and applying the identity of sines to the triangles $OO'a$ and $OO'c$, we have $X : Y = \sin m\theta : \sin n\theta$. The condition that the lunule bounded by the two corresponding circular arcs be squarable in Hippocrates' sense is that $nX^2 = mY^2$, so that $\sin m\theta : \sin n\theta = \sqrt{m} : \sqrt{n}$. If the general case is denoted by the pair (n, m) , then the three cases examined by Hippocrates are $(1, 2)$, $(1, 3)$, and $(2, 3)$. These all happen to be constructible in the familiar "Euclidean" sense, that is, by means of compass and straightedge alone. That two other cases, $(1, 5)$ and $(3, 5)$, are of the same sort was noted by Wallenius (1766) and Clausen (1840)⁸²; for one can readily see that their defining relations reduce to a quadratic in $\cos 2\theta$. Furthermore, researchers have established that the lunule $(1, p)$ is *not* constructible in this manner when p is a prime not of the Gaussian type (that is, of form $2^{2^k} + 1$) (Landau, 1903), nor is the lunule (n, p) so constructible for $n = 1, 2, \dots, p - 1$ (Tschakaloff, 1929).⁸³ Moreover, even for Gaussian prime p (other than 2, 3, and 5), the lunule $(1, p)$ is not constructible (Wegner); furthermore, apart from the cases already known to be constructible, the lunule (n, m) is not constructible when n, m are both odd integers (Tschebotarow, 1935).⁸⁴

For our purposes now, it is important to realize that this modern conception of the problem of constructing the lunules is anachronistic as far as Hippocrates' study is concerned. For we recall that his construction of lunule $(2, 3)$ depended on a *neusis*, that is, the suitable inclination of a line segment of given length. As it happens, an alternative mode for effecting the *neusis* he requires is possible via planar methods (i.e., compass and straightedge), for it may be expressed as a quadratic relation falling within the class of problems known as the "hyperbolic application of areas," so that a solution can be worked out on the model of Euclid's *Data*, Prop. 59, based on *Elements* II, 6 and VI, 29.⁸⁵ While it is a reasonable conjecture that Hippocrates had access to these techniques, nothing in our text directly supports that claim. Certainly, his manner of introducing the *neusis* gives no hint of requiring us to effect this step by any means other than the obvious: the manipulation of the marked line EZB .⁸⁶ Indeed, even later

geometers like Archimedes freely admit *neuses* in their constructions, where sometimes, as in *Spiral Lines*, Prop. 5, an alternative "Euclidean" method is possible.⁸⁷ Thus, I would maintain that Hippocrates' goal was to find any construction he could. If he proceeded from the general conception of the configuration given above, he might perceive that the construction of any lunule (n , m) satisfying the condition $X : Y = \sqrt{m} : \sqrt{n}$ reduces to a determination of the appropriate angle θ . Initially (for $\theta = 0$) the ratio $X : Y$ equals $m : n$. By experience of actually constructing figures, he could realize, if not prove, that as θ increases the ratio decreases, and hence that the specific configuration sought is possible.⁸⁸ In principle, his admission of *neuses* as a constructing technique makes accessible to him a wider range of cases than just those of the "Euclidean" type, including, for example, cases reducing to cubic relations.⁸⁹ Thus, the fact that he presents only the three squarable cases indicates that neither he nor any other ancient geometer who tried his hand at this problem was able to come up with any cases besides these. In this instance, then, not having recourse to the later algebraic methods in geometry constituted a real barrier against the discovery of solutions to these problems.

As noted above, Simplicius assigns greater weight to Eudemos' text than to the one transmitted by Alexander, believing that the former more closely represents the work of Hippocrates himself. Now, from the purely *textual* point of view, he seems quite correct in maintaining this. In certain contexts, for instance, Eudemos adheres to older terminology discontinued in the tradition after Euclid: he speaks of "powers" (*dynameis*) instead of "squares" (*teträgōna*), and uses expressions like "the line *on which* (*eph' hēi*) AB" about as frequently as their short forms (e.g., "the line AB," or simply "the AB"), while only the latter are found in the later formal tradition.⁹⁰ By contrast, Alexander's text is cast in full conformity with Euclidean usage. On the other hand, from the *technical* viewpoint, Alexander's constructions of lunules (a) and (b) are rather simpler than the analogous cases (i) and (iv) of Eudemos; and the technical refinement evident throughout the proofs of Eudemos' four cases clearly surpasses the treatments in Alexander. It is of course conceivable that Alexander's text originated from an effort to simplify an earlier version in the manner of Eudemos. But we recall that in his passage on "reduction," Aristotle alludes to a quadrature set in the form of Alexander's case (b), rather than Eudemos' case (iv). This affirms that Alexander's version too derives ultimately from a 4th-century source. Its simpler treatment of the lunules well suits what one would expect of the early phases of such a study; and as we have seen, it assists considerably in retrieving the motivations underlying the more elaborate constructions given by Eudemos.

This raises the question of the provenance of Eudemos' text. Much depends on Eudemos' reliability and sophistication as a historian. While we possess far too little of his work to make a secure judgment on this matter, it would seem that the purposes occasioning his account of Hippocrates would be admirably served through a fair transcript of his source materials, while Eudemos' motives for producing a radically modified account, indeed even his competence to do

so in such a detailed technical context, would be entirely unclear. Can we then assume that Eudemos' text reflects with reasonable accuracy an actual writing of Hippocrates? There is of course no difficulty in supposing that Hippocrates' initial version of the study of lunules was in the simpler form indicated by Alexander, and that he later produced the more refined version recorded by Eudemos. On the other hand, it is equally plausible that the latter version grew out of the researches of followers inspired by his efforts to pursue this study. Doubtless, Eudemos was in the position to distinguish between a genuine writing of Hippocrates and a later writing in the tradition instigated by him. But would he insist on making this distinction clear in his discussion of the quadratures? We can hope that he would, yet must admit that it is possible he did not.⁹¹

These considerations affect one's use of the fragment as an index of the form and terminology of 4th-century mathematical writing, for we cannot with certainty locate it closer to the time of Hippocrates, toward the beginning of that century, than to the time of Eudemos, nearer its end. Nevertheless, its reliability as a testimony of Hippocrates' findings seems secure, for we have Eudemos' explicit statement several times in the fragment that the four cases of lunules were all studied by him, and it is hard to suppose that he could have been mistaken in this regard. Furthermore, the proofs of the quadratures, even in the simpler version reported by Alexander, indicate advances in the geometric tradition around Hippocrates' time; access to a wide range of theorems and constructions of rectilinear and circular figures, results of the geometric theory of proportions, and so on, as well as appreciation of the basic concerns in the deductive ordering of proofs. But regardless of whether Hippocrates did indeed work out the impressively formalized synthetic treatments given by Eudemos, rather than, say, a more loosely framed analytic treatment, his contribution to the study of these geometric problems is unquestionable.

PROBLEMS AND METHODS

This survey of the earliest efforts within the ancient tradition of problem solving has revealed the central importance of the work of Hippocrates of Chios. In particular, his studies of the cube duplication and of the quadrature of the circle and related figures laid the foundation for later researches. According to Proclus, he was also responsible for the first compilation of "Elements," that is, of a systematized presentation of propositions and proofs.⁹² From the surviving fragments of his work, we may infer that this included substantial materials on the geometry of circles (cf. Euclid's Book III) as well as an exposition of the properties of plane figures dominated by the techniques of proportion theory (cf. Book VI), even if of course there were still lacking the more rigorous methods of limits and proportions (comparable to Books V and XII) due to Eudoxus.

Although stories like the Delian oracle on the doubling of the cube are often cited as suggesting an external motivation for the early studies of such problems, our examination of Eratosthenes' account of them and our inquiry into Hippo-

crates' work serve to discount this view. On the contrary, the interest in these problems can be seen as a fully natural outgrowth of researches within the geometric field. For instance, having cast the problem of squaring a given rectangle into the form of finding the mean proportional between its length and width, Hippocrates could be led to perceive that the finding of two mean proportionals is equivalent to increasing a given cube in a given ratio. In this way, his principal contribution to the study of the cube-duplication problem follows from the same techniques which characterize his study of plane figures. This study of quadratures leads readily to the further question as to whether a comparable construction for the circle might be produced. Here, of course, the quest for a solution fails, but not before the intriguing detour into the construction and quadrature of the lunules. To be sure, only three cases are actually found, although Hippocrates was surely aware of the defining conditions for an unlimited class of quadrable cases. The tantalizing result of having squared a certain lunule in combination with a circle would of course deceive no one into supposing that the circle itself had been squared. But in all these instances an important principle has been established: that curvilinear figures, specifically those associated with circular arcs, are not different in kind from rectilinear figures as far as their quadrability is concerned. Thus, when the later commentator Ammonius (5th century A.D.) presumes to have affirmed the impossibility of the circle quadrature through the intrinsic dissimilarity of the circular and the rectilinear, by analogy with the noncomparability of curvilinear and rectilinear angles, his disciple Simplicius brings forward the quadratures of the lunules as an immediate counterexample.¹

In the matter of the lunules, the standard assumption is that Hippocrates consciously restricted the means allowable in their construction to the "Euclidean" devices of compass and straightedge. Although the cases he presents are indeed constructible by these means, the view flies in the face of his unexplicated introduction of a *neusis* in the construction of his third lunule. While of course the compass and straightedge were by this time long familiar as constructing instruments, and while the range of constructions effectible by these means was doubtless already appreciated to be quite extensive, nevertheless other means were then known, and still others would later be invented. The enterprise of discovering the solutions to problems could hardly be well served by the imposition of such a restriction at this early stage. Hippocrates must surely have been interested in finding constructions, using whatever means were available to him. The explicit restriction to one or another mode of construction is by its nature primarily a formal move, motivated by the urge to divide and classify the collected body of established results. Until the geometric corpus had attained a size and diversity meriting such efforts, there could hardly be much sense in engaging in these formal inquiries. While that level surely was reached around the time of Apollonius, it is open to debate whether it had already reached that level at the time of Euclid.² Applying this notion to the much earlier time of Hippocrates must be viewed as purely anachronistic.

There is a subtle distinction to be made here. The activity of seeking the

solutions to geometric problems is intrinsically formal, to the extent the constructions are to be provided within an implied context that imposes some restrictions on the available means and some measure of deductive ordering in the justifications. Thus, if one can rightly portray the most ancient traditions as empirical in that physical measures might be acceptable to justify certain claimed geometric results, then one cannot assign to these an involvement in problem solving. For in this environment any figure will be squarable, for instance, to within any desired perceptible degree of accuracy. By virtue of addressing the study of geometry in abstraction from overtly empirical measurements, the early Greek geometers would soon come upon configurations whose production proved intractable. For instance, the range of techniques known for the construction and study of similar rectilinear plane figures would be recognized as leading to no evident solution for the cube duplication or the circle quadrature. It is surely no mere coincidence, then, that Hippocrates was the first known systematizer of geometry as well as the first to formulate these constructions as problems, while such an awareness among earlier geometers, even Oenopides, is quite dubious.

On the other hand, the conscious restriction to a specific set of constructing techniques, like the compass and straightedge in sharp separation from others, would be premature at Hippocrates' time. This, I believe, may explain why the problems of rectifying and dividing circular arcs did not seem to attract interest until a century or more later. For a tradition of geometric practice which includes forms of protractors and cords among its tools might readily admit their abstract analogues in the more formalized study of figures, just as the use of sliding rulers gives rise to *neuses* and the use of compasses suggests the Euclidean postulate of circles.³ Thus, the trisection of the angle became a problem only after a tightening of the restrictions on construction techniques, and hence only within a formally more sophisticated geometric field. By contrast, the problems of the cube and the circle present difficulties at the much earlier level of development. Indeed, one may anticipate that it was through the investigation of these and related problems that the clearer conceptions of the general nature of the problem-solving enterprise were to emerge.

NOTES TO CHAPTER 2

¹ Proclus, *In Euclidem*, ed. Friedlein, pp. 283, 333. On Oenopides and Hippocrates, cf. T. L. Heath, *History of Greek Mathematics*, I, pp. 174-202 and the articles in *Pauly Wissowa* and the *Dictionary of Scientific Biography*.

² Heath, *op. cit.*, p. 176; cf. Á. Szabó, *Anfänge der griechischen Mathematik*, III, 19. I criticize this view in my "On the Early History of Axiomatics," p. 150.

³ O. Neugebauer, *Mathematische Keilschrift-Texte*, I (*Quellen und Studien*, 1935, 3: A), pp. 137-142. On the "rope stretchers" see Heath, *op. cit.*, pp. 121ff, and p. 178, where the term appears in connection with the mathematical talents of Democritus.

⁴ This view is advocated with particular zeal by Á. Szabó, as in the work cited in note 2 above.

⁵ Oenopides is reputed, for instance, to have discovered the obliquity of the ecliptic;

see Heath, *op. cit.*, p. 174. This is of course merely a testimony of the infancy of astronomical science among the Greeks in the 5th and 4th centuries, for such facts were certainly long familiar among the Mesopotamians. Of course, the specifically geometric approach to astronomy is characteristic of the Greek style, as contrasted with the arithmetic approach of the Mesopotamians. In this sense, the obliquity might indeed have been a new feature with Oenopides.

⁶ Some of the instruments used in the practical work of geometry and architecture are named by Aristophanes (e.g., *Birds*, 1001 ff) and Plato (e.g., *Philebus* 51c, 56b, 62b). Among later developments one may note the elaborate sighting device described by Hero (in his *Dioptra*, *Opera* III) and the compass for drawing the conic sections devised by Isidore of Miletus (see the interpolation into Eutocius' commentary on Archimedes' *Sphere and Cylinder*, in Archimedes, *Opera*, ed. Heiberg, 2nd ed., III, p. 84). An instrument of the latter kind is described by al-Qūhī and other Arabic geometers; see the account by F. Woepcke, *L'Algèbre d'Omur*, 1851, p. 56n. In the same class may be mentioned the various mechanisms for the cube duplication used by "Plato," Eratosthenes, and Nicomedes (cf. Eutocius, *op. cit.*, pp. 56 ff, 90 ff, 98 ff; these are discussed in Chapters 3 and 6). A survey of the devices familiar in the ancient practical geometry is given by O. A. W. Dilke, *The Roman Land Surveyors*, Ch. 5. On the ancient architectural instruments of stone dressing, in particular, the set square, compass, ruler, and plumb line, see A. Orlandos, *Les Matériaux de construction et la technique architecturale des anciens grecs*, École française d'Athènes: *Travaux et Mémoires*, 16, pt. 2, 1968, pp. 59–69.

⁷ See, for instance, Heath, *op. cit.*, I, pp. 218 ff; Becker, *Mathematisches Denken der Antike*, pp. 74 ff.

⁸ Theon, *Expositio rerum utilium ad legendum Platonem*, ed. Hiller, p. 2. Plutarch, *Moralia* 386e, 579b. These passages have already been cited in Chapter 1.

⁹ Cf. Archimedes, *Opera*, III, pp. 88–96.

¹⁰ See, for instance, Heiberg's note in Archimedes, III, p. 89n; Heath, *History*, I, pp. 244 f; I. Thomas, *Greek Mathematical Works*, I, p. 256n; B. L. van der Waerden, *Science Awakening*, pp. 160 f. They all merely refer to von Wilamowitz as having established the inauthenticity of the document; see note 12 below.

¹¹ Eutocius in Archimedes, III, pp. 88–90 (i), 90 (ii), 90–92 (iii), 92–94 (iv), 94–96 (v).

¹² "Ein Weihgeschenk des Eratosthenes," *Göttinger Nachrichten*, 1894; repr. in *Kleine Schriften* II, 1971, pp. 48–70.

¹³ *Collection*, ed. Hultsch (III) I, pp. 56–58.

¹⁴ Eutocius, *op. cit.*, pp. 90, 96.

¹⁵ See Fig. 1 (a-b) in Chapter 6 for Eratosthenes' construction.

¹⁶ Eutocius, *op. cit.*, pp. 90, 96.

¹⁷ For a gauge of the longevity of such a structure, one may consider the grave marker of Archimedes, all but lost within a century and a half of its foundation; the circumstances of Cicero's rediscovery of the monument in the 1st century B.C. are related in his *Tusculan Disputations*, V, 64ff.

¹⁸ Recall that Hypatia, the learned daughter of Theon of Alexandria, died in 415 A.D. at the hands of a mob of rioting monks; cf. Heath, *History*, II, pp. 528 f, and P. Brown, *The World of Late Antiquity*, London, 1971, pp. 103 f.

¹⁹ See A. Fitzgerald, *The Letters of Synesius*, pp. 258–266 (esp. pp. 262–266); Fitzgerald provides a useful discussion of Synesius' career in the Introduction and notes.

²⁰ Cf., in particular, the prefaces to *Sphere and Cylinder* II and to *Conoids and Spheroids*; those to *Quadrature of the Parabola*, *Sphere and Cylinder* I and *Spiral Lines* provide a certain small amount of contextual information.

²¹ Indeed, G. J. Toomer expresses misgivings over the general assumption that Alexandria was a monopolizing center of technical studies in this period; cf. his *Diocles*, p. 2. One notes that the prefaces to works by Diocles and by Apollonius are somewhat more ample in background information than those to the Archimedean writings.

²² *Opera*, II, pp. 216–258. For accounts, see the Archimedes monographs by Heath, by Dijksterhuis, and by Schneider.

²³ In the 43 pages of the standard edition of the *Sand-Reckoner*, one finds about a dozen second-person forms (e.g., pp. 216, 218, 220, 234, 244, 246, 258); in Eutocius' text of Eratosthenes, a five-page document, there are three (pp. 90, 94, 96), not counting the epigram.

²⁴ For an account of the general background, see M. I. Finley, *Ancient Sicily*, New York, 1968, Ch. 9. I apply these materials toward the dating of the *Sand-Reckoner* in my "Archimedes and the *Elements*," pp. 234–238.

²⁵ Eutocius, *op. cit.*, pp. 88–90; cf. Heath, *History*, I, p. 245. I have inserted the reference letters.

²⁶ On Archytas, see Heath, *op. cit.*, pp. 213–216.

²⁷ Eutocius, *op. cit.*, p. 84.

²⁸ *Ibid.*, p. 96.

²⁹ *Expositio*, ed. Hiller, p. 2.

³⁰ *Moralia*, 386e, 579b–d. Cf. note 8 above.

³¹ *Ibid.*, 718e f; *Vita Marcelli* xiv, 5. The latter contains an extended account of the life and exploits of Archimedes; see the Archimedes surveys cited in note 21 above.

³² Details appear in Chapter 3.

³³ Heath, *History*, I, p. 246; van der Waerden, *op. cit.*, pp. 161–163. The latter seems to treat the story as a dramatization by Eratosthenes; but in a more recent effort, he wishes to treat it as a serious datum reflecting a ritual tradition within early mathematics; see his "On Pre-Babylonian Mathematics (II)," *Archive for History of Exact Sciences*, 1980, 23, pp. 37 f. This thesis is advocated by A. Seidenberg in several articles, particularly, "The Ritual Origin of Geometry," *Archive for History of Exact Sciences*, 1963, 1, pp. 488–527.

³⁴ Eutocius, *op. cit.*, p. 88; to be discussed further below.

³⁵ Note in particular Plato's criticisms of geometric research and his recommendations for mathematical education in *Republic*, Books VI–VII.

³⁶ Cf. *Post. Ana.* I, 7 and *Meta.* M.3. Passages on the ordering and nature of the sciences are collected and discussed by Heath, *Mathematics in Aristotle*, pp. 4–12, 46, 59 f, 64–67, 225.

³⁷ Eutocius, *op. cit.*, p. 88.

³⁸ *Ibid.*, pp. 88–90.

³⁸ A form of the proof, applied to the more general case of similar solids, appears in Euclid's *Elements* XI, 33 (porism).

³⁹ In *Euclidem*, p. 213. On the method of analysis, see Chapter 3.

⁴⁰ *De Anima* II, 2, 413 a 16–20. This passage is noted, if briefly, in most editions of the work; cf. W. D. Ross, Aristotle's *De Anima*, Oxford, 1961, p. 217, and also Heath, *Mathematics in Aristotle*, pp. 191–193. Most commentators, however, attempt to construe it as a specific reference to the Euclidean constructions of either or both of the two problems, rather than as a statement of the reduction of the one to the other.

⁴¹ This fragment, preserved in Simplicius' commentary on Aristotle's *Physics*, is discussed further in this and the next chapter.

⁴² On the ancient Egyptian and Mesopotamian mathematical traditions, see O. Neugebauer, *Exact Sciences in Antiquity*, Ch. 2, 4; and van der Waerden, *op. cit.*, Ch. 1, 2. On the thesis of the ritual origins of technique, see note 32 above. Establishing a perspective on the relation of geometry and philosophy in the pre-Euclidean period is an aim of Chapter 3 below.

⁴³ In *Euclidem*, pp. 422 f. Proclus cites as an example of a theorem on the circle quadrature Archimedes' *Dimension of the Circle*, Prop. 1; see Chapter 5.

⁴⁴ Problem 48; cf. the edition by A. Chace *et al.*, 1927–29, II, Plate 70. For a discussion, see R. Gillings, *Mathematics in the Time of the Pharaohs*, 1972, pp. 139–146, and H. Engels, "Quadrature of the Circle in Ancient Egypt," *Historia Mathematica*, 1977, 4, pp. 137–140. The rule is equivalent to approximating π as $256/81$, that is, $3\frac{13}{81}$ or a bit less than $3\frac{1}{6}$.

⁴⁵ That this relation was recognized is indicated by the use of the correct rule for the volume of the truncated pyramid in the Moscow Papyrus (c. 1600 B.C.); see van der Waerden, *op. cit.*, pp. 34 f. For this would seem to require knowledge of the 1 : 3 ratio between the pyramid and the corresponding parallelepiped. Physical procedures for the measurement of irregular surfaces and solids are found in the Greek metrical tradition; cf. Hero, *Metrika* I, 39 and II, 20.

⁴⁶ *The Birds*, lin. 1001–1005; cf. Heath, *History*, I, pp. 220 f and I. Thomas, *op. cit.*, pp. 308 f. Thomas translates *kampylos kanôn* (literally, "curved ruler") as "flexible rod," and this may be supported in the light of Aristotle's mention of the "leaden rule" used by architects (*Nic. Eth.* 1137 b 30) and by Dioctes' use of a flexible ruler of horn for the purposes of curve plotting; see Toomer, *Dioctes*, pp. 159 f. H. Mendell informs me of an alternative rendering of the passage which conforms better both to sense and to the standard punctuation: "If I lay out the ruler and insert this curved compass from above—do you see? . . . I shall measure with the straight ruler by laying (it) out. . . ." In this way, "curved" modifies "compass" (cf. Aristophanes, *Clouds*, 178) rather than "ruler". This of course does not affect the later testimonia to the use of flexible rulers in geometric practice.

⁴⁷ Cf. Heath, *loc. cit.* The passage itself gives no fully clear notion of just what Meton is up to, and perhaps this is part of the joke.

⁴⁸ *Physics* 185 a 16; *Sophistical Refutations* 171 b 15; cf. Heath, *Mathematics in Aristotle*, pp. 33–36.

⁴⁹ Simplicius, *In Physica*, ed. Diels, I, p. 60. The fragment from Eudemus occupies pp. 61–68, followed by an assessment on pp. 68 f; Simplicius takes up other versions of the circle quadrature on pp. 53–60.

⁵⁰ *Collection* (V), I, pp. 340–342. See my discussion in "Infinity and Continuity," pp. 127 ff.

⁵¹ On this terminology, see my "Archimedes and the *Elements*," pp. 240n, 254n, 264; and my "Archimedes and the Pre-Euclidean Proportion Theory," p. 196.

⁵² On the Eudoxean methods and their attribution via Archimedes' *Quadrature of the Parabola* and *Sphere and Cylinder* I, see my "Archimedes and the Pre-Euclidean Proportion Theory," pp. 194 ff.

⁵³ For a general account of Antiphon's thought, see W. K. C. Guthrie, *A History of Greek Philosophy*, III, pp. 285–294. Simplicius' passage on his circle quadrature (*In Physica*, pp. 54 f) would appear to derive from Eudemus, for his specific critique of this argument is cited (*ibid.*, p. 55). The same critique, if without express citation of Eudemus, is given also by Themistius (4th century A.D., *In Physica*, ed. H. Schenkl, CAG 5, Pt. II, 1900, pp. 3 f), namely, that it violates the principle of the unlimited divisibility of continuous magnitude. But Simplicius also cites an alternative critique by Alexander, so that he may well have derived his account of Antiphon via Alexander, rather than directly from Eudemus.

⁵⁴ Simplicius, *op. cit.*, pp. 54 f.

⁵⁵ *Ibid.*, p. 55 (see note 53 above). On Protagoras' treatment of this issue, see Aristotle's *Metaphysics* B, 2, 997 b 32 and its discussion by Guthrie, *op. cit.*, II, p. 486 and III, p. 267. On Democritus' work, see Heath, *History*, I, pp. 176–181 and Guthrie, *op. cit.*, II, Ch. viii, especially pp. 487 f.

⁵⁶ Cf. Eudemus' view, cited in note 53 above.

⁵⁷ See, for instance, Heath, *History*, I, p. 224.

⁵⁸ See my "Infinity and Continuity," pp. 133 f. The summation theorem for the finite case is given in *Elements* V, 12.

⁵⁹ Hippocrates seems to appeal to a concept of proper "parts" as the basis of his notion of ratio; cf. Simplicius, *op. cit.*, p. 61 and the discussion by Heath, *History*, I, pp. 187–191. This could provide the precedent for a theory of proportions for commensurable magnitudes; but the extension to incommensurable magnitudes, as in the Eudoxean theory, had not yet been worked out at Hippocrates' time. See my "Archimedes and the Pre-Euclidean Proportion Theory," and Chapter 3.

⁶⁰ Further examples are found in Pappus' measurements of figures bounded by spirals; see Chapter 5.

⁶¹ Plutarch, *Moralia* 607 f (cf. Diels and Kranz, A 38, II, p. 14). The view that this might signify a writing by Anaxagoras on circle quadrature is discounted by J. Burnet (*Early Greek Philosophy*, 4th ed., p. 257) and by Guthrie (*op. cit.*, II, p. 270).

⁶² In *Euclidem*, pp. 65 f. Vitruvius also mentions Anaxagoras for a study of perspective; cf. Heath, *History*, I, pp. 172–174. For a general account of Anaxagoras, see Guthrie, *op. cit.*, II, Ch. iv.

⁶³ *Prior Analytics* II, 25; cf. the passages cited in note 48 above.

⁶⁴ See note 49 above.

⁶⁵ Simplicius, *op. cit.*, pp. 56 f.

⁶⁶ This readily suggests an alternative construction in which the right triangle may be scalene rather than isosceles. Here too the two lunules will sum to the area of the triangle.



Figure 8

* This is done by applying the principle that in right triangles the square on the longest side equals the sum of those on the other two sides, but in obtuse triangles it is greater and in acute ones less; cf. *Elements* I, 47 and II, 12-13. In the present case, since angle BAG is obtuse, $BG^2 > BA^2 + AG^2 = 2 GD^2$, whence $BD^2 (= 3 GD^2) < BG^2 + GD^2$, so that angle BGD is acute. Hence, arc BGD is greater than a semicircle, since the angles inscribed in segments are proportional to the corresponding supplementary arcs (cf. Simplicius, *op. cit.*, p. 60).

²⁹ See Heath, *Archimedes*, pp. ci-ciii, who follows Zeuthen in supposing that Archimedes must have understood alternative constructions of the *neuses*. But Dijksterhuis (*Archimedes*, pp. 138 f) insists, I believe correctly, that Archimedes accepted the *neuses* in their own right; see Chapter 5.

²² As θ continually decreases from $180/(m+n)$ to 0, the ratio $X:Y$ continually increases from 1:1 to $m:n$. Since the ratio $\sqrt{m}:\sqrt{n}$ lies within this range, $X:Y$ will assume this value for some angle within the corresponding domain of θ . One may observe that Hippocrates' *neusis* for the case (2, 3), when $X=Y$, yields a construction for the regular pentagon. Some have wished to assign a method of this sort to the ancient Pythagoreans; but, unfortunately, there is no documentary support for this claim (cf. R. Böker, "Winkelteilung," pp. 137 ff.).

²³ In this connection, Steele notes Vieta's treatment of the case (1, 4); see *op. cit.*, p. 318. *Neuses* may be used for the solution of third-order problems like the cube duplication and the angle trisection; see Chapters 5, 6, and 7. For an account of this construction technique, see L. Bieberbach, *Theorie der geometrischen Konstruktionen*, Basel, 1952, Ch. 16, 17.

²⁴ See note 70 above. The language of proportions here is also archaic. Eudemos' text speaks of the proportion $A:B = G:D$ by the expression "A is greater than B by the same part that G is greater than D" (Simplicius, *op. cit.*, p. 60). This may be compared with the phrasing in Archytas (fr. 2; Diels-Kranz, 47 B 2, 1, pp. 435 f) and in the pseudo-Aristotelian *Mechanics*, Ch. 20. By contrast, Euclid, Archimedes, and other late writers will say "A has the same ratio to B that G has to D," or more simply, "A is to B as G is to D."

²⁵ Note, for instance, that the commentators commonly speak of "those in the circle of . . ." (*hai peri*), thus leaving it unclear whether they refer to the master of the tradition or to his followers. Eudemos himself was evidently willing sometimes to make historical claims on the basis of inference. He says of Thales, for instance, that he knew a certain theorem on congruent triangles (*Elem.* I, 26), for he "must have used this" for determining the distance of ships at sea, as he is reported to have done (cf. Proclus, *In Euclidem*, p. 352). The difficulty of discerning the actual basis of this and other claims by Eudemos has thus led to a spectrum of widely divergent views on the achievements of Thales. For references, see W. Burkert, *Love and Science in Ancient Pythagoreanism*, 1972, pp. 415-417.

²⁶ Proclus, *In Euclidem*, p. 66.

²⁷ Simplicius, *op. cit.*, pp. 59 f. Ammonius here refers to the "horn angle" defined by the space between the circle and its tangent (cf. *Elem.* III, 16 and Heath, *Euclid*, II, pp. 39 ff). Proclus makes several references to this and related mixed and curvilinear angles (cf. *In Eucl.*, pp. 122, 127, 134). These figure within Philoponus' discussion of Bryson's circle quadrature (see Chapter 3).

²⁸ This will be discussed in more detail in Chapters 7 and 8.

²⁹ In particular, the availability of "curved (flexible) rulers" suggests an extremely simple practical procedure for these constructions; see note 46 above and Chapter 3.

CHAPTER 3

The Geometers in Plato's Academy

Through Hippocrates' efforts a start was made toward the investigation of geometric problems. But the first discovery of actual solutions for more difficult problems, like the circle quadrature and the cube duplication, was made by geometers in the generations after him. Hippocrates' reduction of the latter problem, for instance, to the form of finding two mean proportionals between two given lines served as the basis for the successful constructions by Archytas, Eudoxus, and Menaechnus. At the same time, new geometric methods were introduced, in particular the use of special curves generated through the sectioning of solids or through the geometric conception of mechanical motions. Furthermore, Hippocrates' precedent for the "reduction" of one problem to another could lead others to the articulation of the versatile technique of geometric "analysis."

During the 4th century B.C., Plato's Academy in Athens became a center for geometric studies. This is reflected, as we have seen, in such legendary accounts as that of the Delian oracle, portraying Plato's interaction with mathematicians. Although later authorities sometimes assign to him the discovery of specific technical results, like a method for the duplication of the cube, the veracity of such reports is dubious and generally discounted. In the reasonable assessment by Proclus, it was rather through the special position he afforded mathematical study within his program of philosophical education, elaborating a plan initiated by the older Pythagoreans, that Plato encouraged mathematical research, as well as through his incorporation of technical examples into his writings to illustrate points of method.¹ Plato appears to have been much impressed by the technical rigor of some of the older geometers, notably Theodorus of Cyrene, a contem-

porary of Socrates, and by the genial insights of geometers of his own generation, in particular the Pythagorean Archytas of Tarentum and Theaetetus of Athens.² Of other geometers, like Leon and Leodamas, we know little. But towards the middle of the century, Eudoxus of Cnidus became affiliated with the school, and he and his immediate disciples, like the brothers Menaechmus and Dinos-tratus, profoundly stimulated research into mathematics and mathematical science. It is natural to suppose that the formal precision characteristic of the geometric methods of Eudoxus developed in part as a response to the intellectual emphases of this philosophical environment. Interestingly, Eudoxus' tenure at the Academy coincided with that of the young Aristotle, so that the work of this geometer can hardly but have had its influence on the elaboration of the philosopher's views on logic and the formal structure of science.³

Recognizing these aspects of the context of research in this period will prepare us for its distinctive character. Even as the geometric field expanded impressively in the range of established technical results, there advanced parallel to this a rigorization of the forms of mathematical exposition. The latter has often been exploited as evidence of the influence of philosophical developments on mathematical research and has induced many scholars to propose dialectical reasons for explaining technical issues, like the ancient treatment of infinity or the selection of methods of construction. But such views are too readily exaggerated. Surely, the technical disciplines were on the whole autonomous in setting the directions of research. If in this period geometry and philosophy interacted to a greater degree than at other times in antiquity, the effect on geometry was subtle and not specific: an enhanced sensitivity to questions of formal proof. By contrast, the reciprocal effect on philosophy could often be specific, as mathematics offered clear models for inquiries into the general structure of knowledge. Failure to appreciate this distinction must inevitably hinder the effort to capture the nature and motives of technical work in the pre-Euclidean generations.

SOLUTIONS TO THE CUBE DUPLICATION

According to Eratosthenes, three of Plato's associates worked out solutions for the Delian problem: Archytas "via semicylinders," Eudoxus "via curved lines," and Menaechmus "via the triads cut out from the cone."⁴ Among the dozen methods in his survey, Eutocius presents some remarks on a text of Eudoxus' method, along with complete details of the construction and proof of those of Archytas and Menaechmus, the former based on an account by Eudemus. In the present section we review these, adding a fourth method transmitted by Eutocius under the name of Plato.

The construction of Archytas is a stunning *tour de force* of stereometric insight.⁵ Its essential idea, following the form of the reduction of the problem by Hippocrates, is to find the two mean proportional lines AI , AK between two given lines AM , AD via an arrangement of similar right triangles as shown in Fig. 1. Archytas determines point K as the intersection of three solids. Starting with a semicircle whose diameter AD' equals the larger given length and in

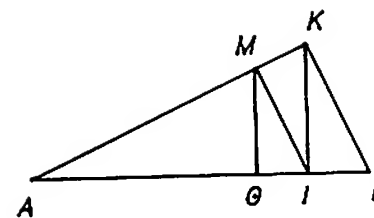


Figure 1

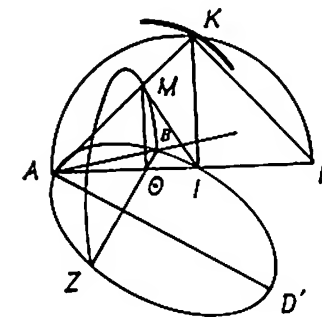


Figure 2

which the chord AB equals the smaller, he conceives an equal semicircle erected in the plane perpendicular to the given semicircle: as the diameter AD' turns on the pivot A , this circle will generate the surface of a torus (see Fig. 2). If a right semicylinder has been erected over semicircle ABD' as its base, then the torus will meet it along a certain curve. At the same time, he conceives that the given semicircle rotates about its diameter AD' as an axis, so that the extended chord AB generates the surface of a cone. Let this intersect the "cylindric curve" at K . Archytas then shows that this effects the construction: if one draws triangle AKD , the line KI perpendicular to AD , the line AM equal to AB , and the line MI perpendicular to AD , he can prove that the angle AMI is a right angle. Hence, the three triangles AMI , AIK , AKD are similar, so that the four lines AM , AI , AK , AD are in continued proportion.

The synthetic expository mode here adopted by Eutocius inevitably obscures the underlying pattern of thought. But an analysis along the following lines may suggest an approach available to Archytas. Since the objective is to find an arrangement of the triangle AKD such that AM has the given length, let the chord AK be conceived to rotate on A , and for each position of K let there be drawn KI perpendicular to AD and IM perpendicular to AK (see Fig. 3). Then M will trace a curve such that the ray AM continually decreases from its initial value AD , assuming arbitrarily small values in the vicinity of A . At some position, then, AM will equal the given line AB , and from this the solution of the problem follows.⁶ Instead of effecting this via a pointwise construction of the locus of M , one can seek the common intersection of the loci associated with the points K , I , and M .⁷ The locus of K , as the vertex of a right triangle of hypotenuse AD , is secured by keeping it on the circumference of the circle AKD (see Fig. 4). The length AM can be kept equal to AB by setting M on the trace of B as its semicircle rotates on its axis AD . As for the locus of I , this can be solved as follows: in the desired terminal position, we require that IM be perpendicular to AK , where I is figured as the foot of the perpendicular from K to AD . Now, M lies on the circle ZMB of diameter ZB : setting θ as the

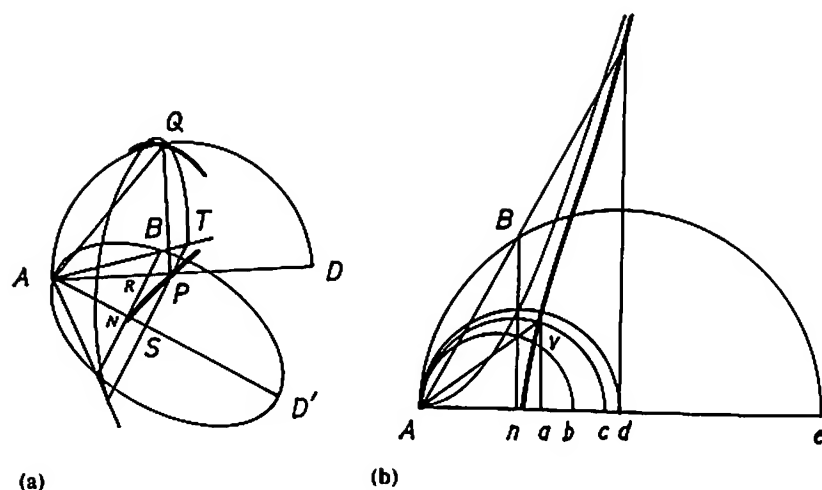


Figure 6

: $AN = AP : AS$. But by similar triangles, $AP : AS = AR : AN$, so that $AR : AN = AS : AN$, or $AR = AS$. This last result leads to the construction of the curve as specified above, where one must introduce some form of pointwise procedure actually to produce the curve and find its intersection with the semi-circle ABD' .¹¹

A proposal of quite a different sort has been made by R. Riddell, who links the recursive configuration of right triangles essential to Archytas' method with a kinematic configuration derived from Eudoxus' work in geometric astronomy.¹² Addressing the problems of describing the motion of planets by means of uniform circular motions, Eudoxus hit upon the conception of the "hippopede" curve: if one considers a point P set on the equator of a rotating sphere, and one embeds this sphere within a second sphere rotating with an equal speed in the opposite direction and communicating its own motion to the inner sphere, then if the two spheres have the same poles, the point P will merely remain fixed; but if the spheres are inclined at an angle, P will trace a curve in the form of a double loop, or figure-8, for each complete revolution of the two spheres. (See Fig. 7a.) It was named "hippopede," or "horse fetter," for its shape, and may be alternatively conceived as the intersection of the inner sphere with a right cylinder.¹³ Projected onto the equatorial plane of the outer sphere, the curve becomes the circle ARH [see Fig. 7(b)]. The effect of the rotation of the inner sphere is to raise P above this plane from its initial position at A , so that its projected position is on the circle $QR'G$; the effect of the rotation of the outer sphere is to transfer the whole figure back over its initial position ARH , thus leaving the projected position of P at R .¹⁴ If this looping path of P is now superposed over the slow rotation of a point on the equator of yet another enclosing sphere (corresponding to the ecliptic circle on the celestial sphere), then the resultant

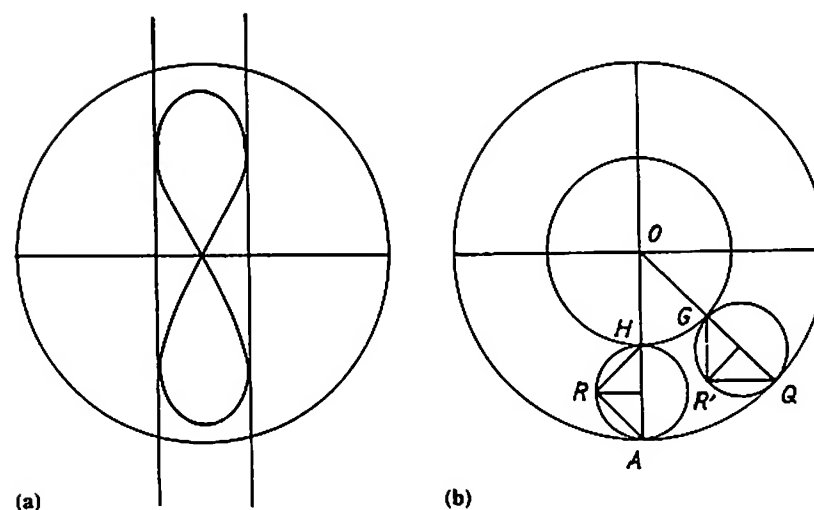


Figure 7

motion of P gives a fair semblance of the path which planets are seen to mark off in the heavens. Unfortunately, this model proves remarkably inept at saving the observed appearances of planetary motions: it fails to account for the variations in orientation, size, and shape of successive loops; in some instances, like Venus, it does not yield a retrograde motion at all; and it places the planet always at the same distance from the observer and so leaves unexplained the differences in brightness.¹⁵ Surely, such difficulties as these induced later astronomers to try entirely different approaches, like the heliocentric model of Aristarchus, setting the planets each in its own circular orbit around the sun and each with its own period of revolution.

But if the Eudoxean scheme of homocentric spheres held limited interest for astronomers beyond a few decades after Eudoxus, this need not signify that the curve he introduced failed to draw the attention of geometers. Indeed, the hippopede is mentioned several times by later authors for one or another of its features.¹⁶ Riddell thus invites us to consider another curve generated via a modification of Eudoxus' arrangement. Instead of having the outer sphere rotate in reverse with the same speed as the inner, let it rotate with twice that speed. This will have the effect of returning the projected position of $QR'G$ through an equal angle to the opposite side of its initial position, so assuming the position JRH [see Fig. 8(a)]. The space curve traced by P will thus arch to an upward maximum corresponding to the first quarter-turn of the inner sphere, then move downward, arriving at its minimum at the end of the third quarter-turn.¹⁷ The curve of projection will accordingly pass through two inward-facing arches, as it weaves within the space bounded by an outer circle and an inner ellipse [see Fig. 8(b)]. Now consider the projection R of an arbitrary position of P [see Fig.

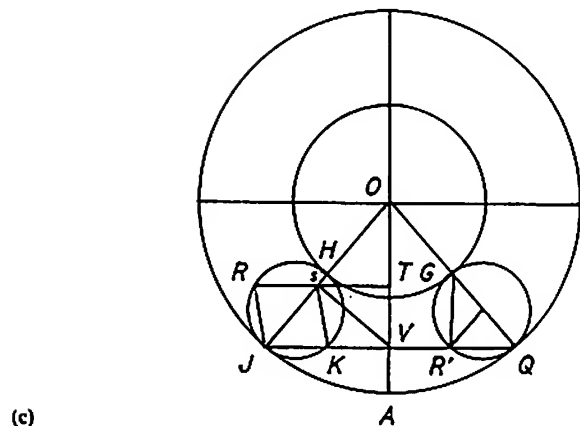
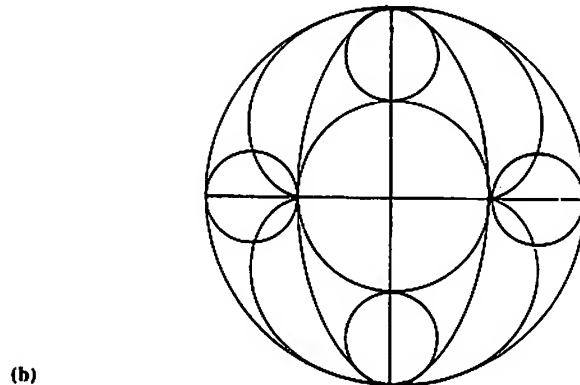
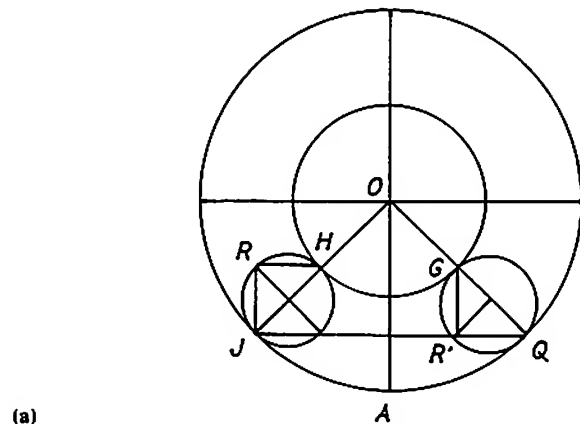


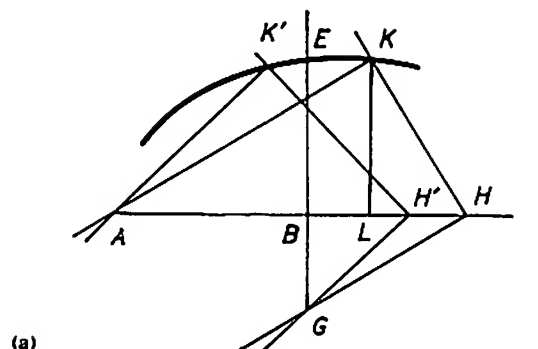
Figure 8

8(c)). Draw the line RT perpendicular to the initial radius OA, and meeting the terminal radius OJ at S. Since the angles at O are equal, JQ is perpendicular to OA, meeting it at V. Drawing SV, Riddell shows that VS will be perpendicular to OJ on the condition that OH is one-half OA.¹⁸ For this makes R' the bisector of QV, whence JK (side of the rhombus RSKJ, where RJ = R'Q) will equal one-half JV; hence, VK = KJ = KS, so that angle JSV is a right angle. Thus, the triangles OTS, OSV, OVJ fall into the configuration central for Archytas' duplication of the cube: that is, the lines OS, OV are two mean proportionals between OT, OJ. Thus, to find the two means between two given lines, one produces the projection of this modified Eudoxean curve, where OA equals the larger given line and OH equals one-half of the line; marking off the shorter line as OT, one then draws the perpendicular to OA at T, meeting the curve at R, and one locates the point R' as the projection of the point P' where the circle through P parallel to the equator of the outer sphere meets the equator of the inner sphere. Then, the perpendicular to OA from R' meets the outer circle at J and OA at V, from which the diagram may be completed to yield OS, OV as the required mean proportionals.¹⁹

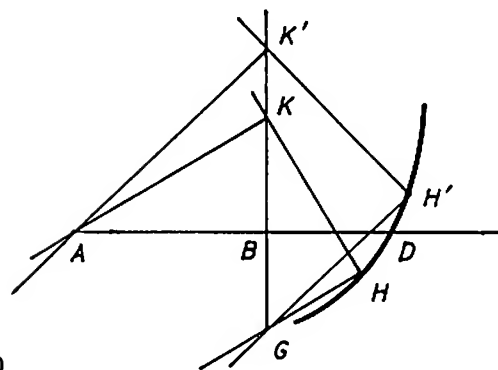
Relative to these proposals by Tannery and Riddell, it is a mere quibble to point out that Archytas does not actually introduce the section of the torus and the cone, nor does the ancient tradition ever signify such a variant on Eudoxus' hippopede.²⁰ These are both ingenious constructions, quite in the spirit of the ancient geometrical methods. It will emerge from what follows, however, that further considerations point to a different alternative.

The construction "according to Plato" is the first of Eutocius' texts on the cube duplication, and thus follows immediately his brief comment on the omitted Eudoxean method.²¹ The "Platonic" procedure employs a mechanical device to produce a configuration consisting of three consecutive similar right triangles GBD, DBE, EBA, where the sides AB, BG are given [see Fig. 9(a)]. Its base is the beam H0 pivoting on the given point G such that its endpoint H always lies along the line of AB [see Fig. 9(b)]. A second beam HZ is fixed at a right angle to the first. A third beam KL is free to slide along HZ, but is always perpendicular to it. If K is set along the line of GB, then KL will meet BA (or its extension), say at A', and when this point comes to coincide with A in the course of the device's manipulation, the construction will have been effected, so that DB, BE will be two mean proportional lines between the given lengths GB, BA [see Fig. 9(c)].

The Platonic provenance of this method is open to serious question. First, Eratosthenes makes no reference to it among his remarks on the early efforts at cube duplication. This is all the more noteworthy since his interest in the Platonic philosophy is evident through his writing of the *Platonicus* and his elaboration there of the story of Plato's involvement with the Delian oracle.²² Surely, then, Eratosthenes would have been particularly keen to indicate an actual solution by Plato of the Delian problem, had he known of one. Furthermore, this method depends on the conception of a mechanical device. Indeed, Eutocius' text goes into considerable detail on the fashioning of the grooves and the pivots ensuring that the beams maintain their perpendicular orientations throughout their motion.



(a)

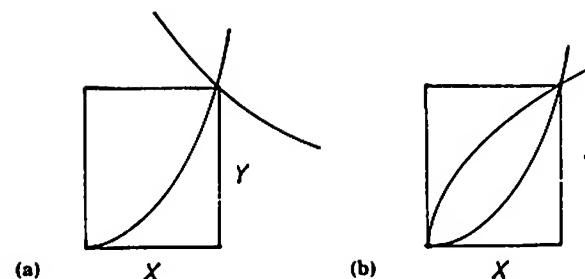


(b)

Figure 10

geometric manner just given. Neither actually generates a curve, although the method for doing so is evident. The omission might well be due to Menelaus, for he does the same thing in his treatment of Archytas' method.³⁷ We thus have a possible explanation of how a text of Eudoxus' solution "by means of curved lines" might come into Eutocius' hands, now bereft of any mention of curves. Moreover, if the text was at all unclear on the positioning of K along line GB, one would not see how the discrete proportion $GB : BH = HL : LK$ becomes the continuous proportion $GB : BD = BD : BK$. These two defects marred Eutocius' source on the Eudoxean method and so suggest an association with the pseudo-Platonic method.

Under this view of the identity of the two methods, we would have to assign to Eudoxus a treatment in the more abstract manner suggested above, for Eratosthenes criticizes Eudoxus, no less than Archytas and Menaechmus, for having addressed the problem in a strictly theoretical manner without thought of practical execution. As indicated above, Eratosthenes himself emerges as a likely candidate to bear responsibility for casting this method into the form of an actual



(a)

(b)

Figure 11

mechanical device. His writing of the *Platonicus* would offer a natural context for this and through this association induce the later writers mistakenly to attribute the method to Plato. One can detect in Eutocius' version of the pseudo-Platonic method certain coincidences of phrasing which link it to other texts he presents.³⁸ Such editorial interventions on his part would be necessary in the event, as we propose here, that his source ultimately derived from the relatively informal context of a discussion of the Platonic philosophy by Eratosthenes.

The third of the early solutions mentioned by Eratosthenes entailed the "sectioning off from the cone the triads of Menaechmus." Eutocius presents a text of Menaechmus' method in which the desired lines are obtained first from the intersection of a hyperbola with either of two parabolas, then, in a second version, via the intersection of the two parabolas.³⁹ More specifically, if we wish to find for two given lines A, B the two lines X, Y which are the mean proportionals between them, then we may construct the parabola satisfying the condition $X^2 = AY$ (or, indifferently, the parabola satisfying $Y^2 = BX$) and the hyperbola satisfying $XY = AB$. The intersection of these two curves yields the lines X, Y which are the desired mean proportionals [see Fig. 11(a)]. Alternatively, we may consider the intersection of the two parabolas; for this gives rise to the same two lines X, Y [see Fig. 11(b)].

The essential line of thought, which we have paraphrased here following Eutocius, is entirely clear, especially since Eutocius precedes his formal synthesis of the problem with an analysis. Assuming the four lines have been set out, such that $A : X = X : Y = Y : B$, if X is set at a right angle to Y, the proportion $A : X = X : Y$ associates X and Y as coordinates of a given parabola; similarly in the instance of the hyperbola and the second parabola. The notion of setting the lines at right angles is quite naturally suggested by the configuration of the pseudo-Platonic method which would surely be familiar to Menaechmus, if that was indeed the method used by Eudoxus [see Fig. 11(c)]. What has puzzled scholars is the manner by which Menaechmus was able to construe these relations as properties of sections of cones. Eutocius' text is of limited use in this regard, for it is framed entirely in conformity with the Apollonian form of the theory

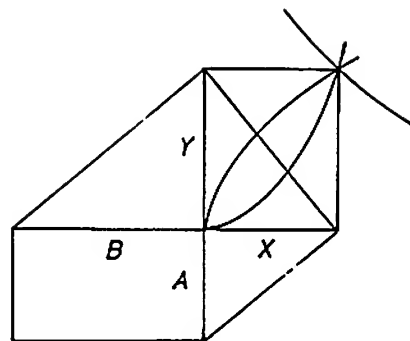


Figure 11

of conic sections. A text adhering closely to the treatment of this problem by a 4th-century geometer like Menaechmus could hardly employ terminology like "latus rectum," "ordinate," "asymptote," and perhaps not even the terms "parabola" and "hyperbola." Thus, modern commentators have engaged in ambitious reconstructions of the rudimentary theory of conic curves which Menaechmus might have developed for the purposes of his solution of the cube duplication.³¹ As one can well anticipate, any such theory must have embraced a substantial portion of the content, if not of course the form, of Apollonius' Books I and II.

This effort has provoked surprisingly little uneasiness among historians of this period.³² Yet apart from this testimonium of the work of Menaechmus, there are but two passages which might reflect the very early study of conics, the one in the pseudo-Aristotelian *Problems*, the other in Euclid's *Phaenomena*, and hence both datable to around 300 B.C.³³ At about that same time Euclid and his contemporaries were in the position to make a first compilation of the theory of conics, of which we possess a good, if indirect, indicator in the works of Archimedes.³⁴ But the tradition of an early theory instituted by Menaechmus nearly a half-century before Euclid depends on but a single line of Eratosthenes' verse: "Do not thou seek to section off from the cone (*kōnotomein*) the Menaechmean triads."³⁵ Reading this line in its context, we recognize at once that the "Menaechmean triads" (note the plural) refer to the three specific conic curves (the hyperbola and two parabolas) associated with each choice of the two given lines in the problem of finding the two mean proportionals. Contrary to the familiar view, these can hardly refer to the three forms of conic section: parabola, hyperbola, and ellipse. For at no known stage of the ancient theory were these curves generated together as triplets, while the ellipse has no bearing on the ancient solutions of the cube duplication, as transmitted to us, whether from Menaechmus or from anyone else. Eratosthenes' term *kōnotomein* is no

guarantee that Menaechmus actually produced these curves as sections of cones, for anachronistic applications of terminology have been known to occur among historians of mathematics, while Eratosthenes' epigram was not composed as a historical effort anyway. Indeed, strictly speaking, Eratosthenes does not even claim that Menaechmus sectioned the cone, but only that "you," the contemporary reader of the epigram, need not do so. We thus have good cause to consider whether Menaechmus might have constructed his solving curves in some other manner.

An appropriate method is readily conceived on the basis of the pointwise construction of the curves answering to the three relations derived from the assumption of the continued proportionality of the lines A, X, Y, B .³⁶ In the operation of the means-finding apparatus attributed to Plato (Fig. 9b), we maneuver the right angle GHZ so that the leg GH always passes through G while H always lies on the extension of AB [Fig. 12(a)]; if for each position H we complete the rectangle $HBKN$, the vertex N will trace a curve whose coordinates are such that $BH (= X)$ is the mean proportional of $GB (= A)$ and $BK (= Y)$. Similarly, if the right angle AKH is turned so that the leg AK always passes through A while K lies on the extension of GB [Fig. 12(b)], completing the rectangle $HBKX$, we trace the curve whose coordinates are such that BK is mean proportional of HB and $BA (= B)$. Thus, where these two curves intersect, the coordinates X, Y will be the two mean proportionals between the given lines A, B , as required by the problem. In an alternative construction, we can figure Y as the half-chord perpendicular to the diameter of a circle such that Y divides that diameter into segments of length B and X (cf. *Elements* VI, 13 or II, 14 [see Fig. 12(c)]). To produce the curve associated with the relation $Y^2 = BX$, then, one sets out as many lengths for X as one wishes, determines Y for each in the above manner, locates a point for each such pair of coordinates, and connects the points by means of a flexible ruler or other suitable drawing aid. The construction of the curve associated with the relation $X^2 = AY$ is of course the same [see Fig. 12(d)]. In the case of the relation $XY = AB$, one may find Y coordinate to X via the "parabolic application of area," based on the fact that the diagonal of the rectangle of sides $B + X$ and $A + Y$ passes through the origin (cf. *Elements* I, 44) [see Fig. 12(e)]. Otherwise, the generation of a curve through a selection of values for X follows the same pattern. Having drawn two of these curves over the given lines A, B , one determines from their point of intersection the lines X, Y which solve the problem of the mean proportionals.

Pointwise procedures of this sort are familiar among later writers. Diocles, for instance, constructs his own curve for solving the cube duplication by determining a set of points and then connecting them via a flexible ruler; his alternative construction of the parabola, framed in effect around its focus-directrix property, is also in the pointwise manner.³⁶ Eutocius describes analogous procedures for drawing all three kinds of conics, and notes that writers on mechanics often adopt such methods for the conics where other methods do not work well.³⁷ Similarly, Anthemius marks out the contours of burning mirrors by orienting the tangent lines at selected points.³⁸ No source identifies the originator of these

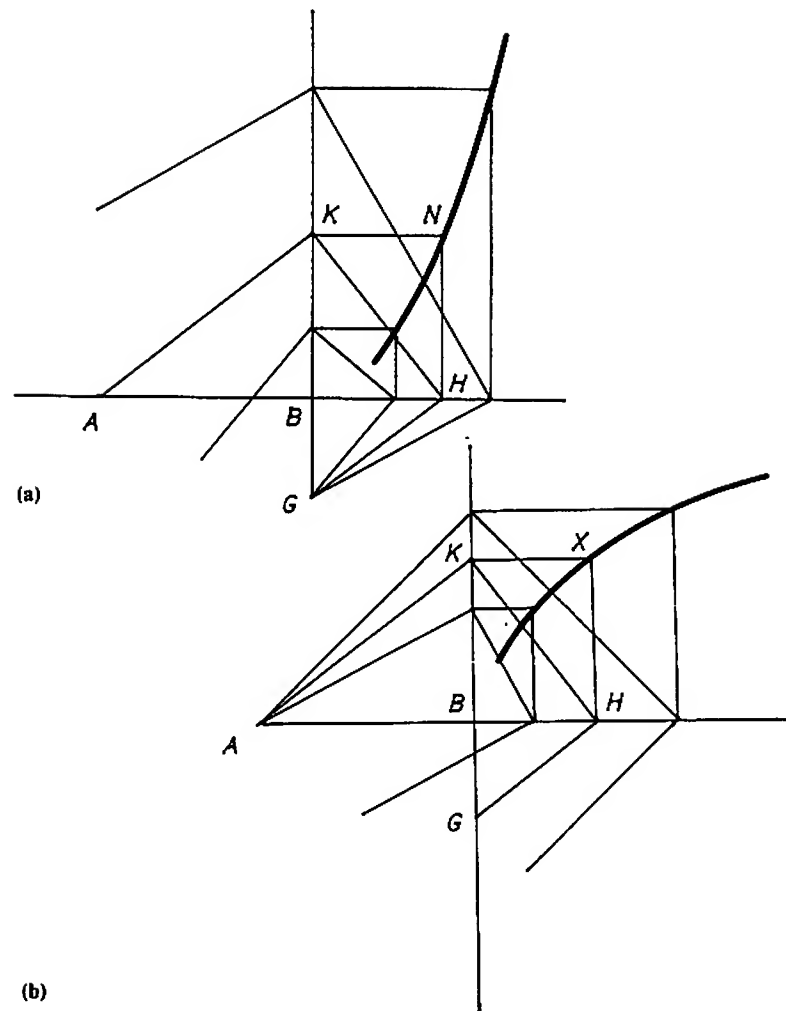
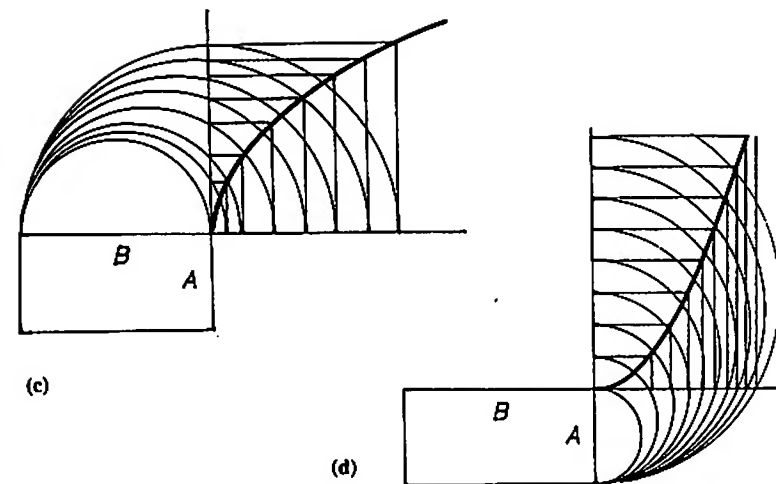


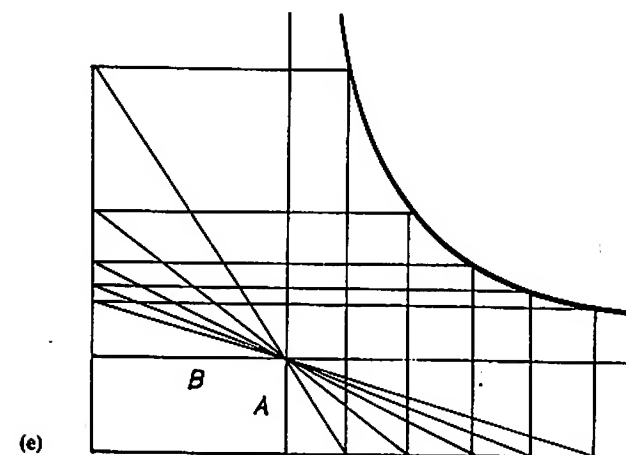
Figure 12

pointwise methods. But Diocles' dependence on Menaechmus is evident in his presentation of a form of the two-parabola solution for the cube duplication, and one can readily assume that he also found with Menaechmus a precedent for his use of a pointwise construction of those curves.³⁹ It is right, I believe, to view this method of constructing curves to be an insight of significance in the development of geometry, yet hardly overreaching the abilities of pre-Euclidean geom-



eters. Indeed, its invention prior to Menaechmus, or perhaps by him, is rendered plausible by the association with Diocles.

I propose, then, that Menaechmus based his solution on curves defined with respect to the second-order relations among the mean proportional lines. This does away with the warrant to speculate on how Menaechmus might have formed these curves as sections of cones and which of their properties he might have



proved, exercises which have all too long distracted historians from considering evidence from around the time of Euclid, a more plausible context for the first study of these sections. This view has the welcome corollary of restoring confidence in the reliability of Eutocius, our principal source of documentary materials on this phase of early geometry. Although his account of Menaechmus unquestionably reflects the reworking effort of a post-Apollonian editor, it serves now as an accurate guide to the essential line of thought Menaechmus followed.

GEOMETRIC METHODS IN THE ANALYSIS OF PROBLEMS

The set of 4th-century solutions to the cube duplication already exploit a wide range of geometric methods which would remain characteristic features of the ancient problem-solving tradition. Each of the methods considered, for instance, introduces one or more special curves, generated either through the section of solids, as in Archytas' solution, or through the mechanical motion of line segments, as in the pseudo-Platonic arrangement, or through pointwise constructions, as proposed for the solution by Menaechmus. Eutocius' text of Menaechmus' solution highlights two further aspects of methodology not displayed in any of his other ten texts. One is its use of the second-order relations of lines associated with the pre-Euclidean technique of "application of areas," on which was built the later theory of the conic sections. The second is its adoption of the two-part format of "analysis and synthesis" which would become, in the hands of Apollonius and his contemporaries, a powerful instrument for the discovery and exposition of the solutions to problems.

The "application of areas," as we learn from Proclus on the authority of Eudemus, owes its origins to the ancient Pythagoreans, although Plutarch knows of a tradition assigning its invention to Pythagoras himself.⁴⁰ It relates to a series of problems now extant in Euclid's Book VI and supporting lemmas to be found in his Books I and II. The objective is to "apply" (*paraballein*) a given area to a given line segment (I, 44) or, alternatively, to construct a parallelogram similar to a given figure and equal to a second given figure (VI, 25). One may also seek to apply the given area to a given line segment such that the resulting parallelogram either "exceeds" (*hyperballein*) or "falls short" (*elleipein*) by a figure similar to a given parallelogram (VI, 29, 28, respectively; cf. II, 6, 5). In the earlier stages of this technique, one would take the figures of application to be rectangles and the figures of excess or defect to be squares. Thus, for the first, or "parabolic" case, one seeks to construct the line X such that $AX = M$ for a given line segment A and a given rectilinear area M , e.g., the square of side B . In the "hyperbolic" case, one seeks X such that $X(X + A) = B^2$, and in the "elliptic" case, the X such that $X(A - X) = B^2$ [see Figs. 13(a), (b)].⁴¹ The attribution to the Pythagoreans merely confirms what one infers from Euclid's elements: that these techniques were familiar in the pre-Euclidean geometry, that is, in the 4th century B.C. and perhaps the latter part of the 5th century B.C. but it is now well known that the same problems, treated in the form of numerical

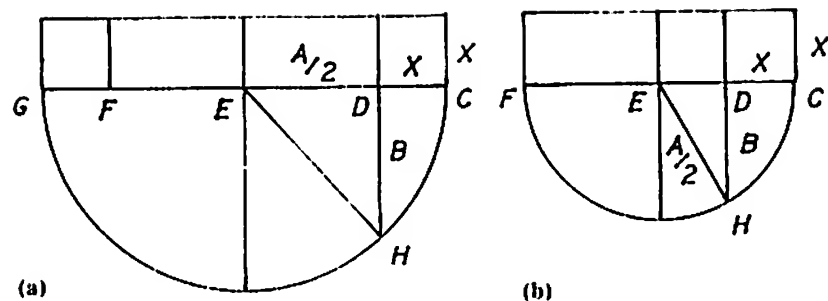
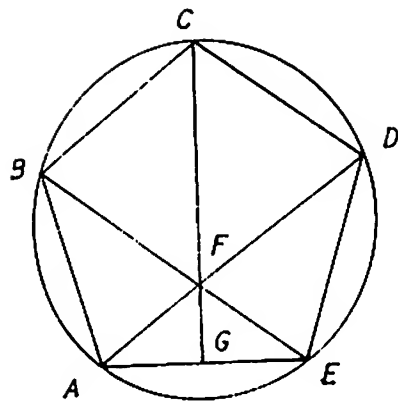


Figure 13

solving procedures, were a prominent part of the Babylonian tradition more than a millennium earlier.⁴² One would thus infer that the Greeks first learned these techniques through contact, direct or indirect, with the Babylonian tradition.⁴³ The same would also have provided them a precedent for working out ordered procedures for the solution of *problems*; but the specifically *geometric* form into which the Greeks cast their own version of this technique is not manifest in the Babylonian texts now extant.

A most notable example of the "application" technique is in the division of a line into "extreme and mean ratio" (VI, 30; cf. II, 11), through which one can effect the inscription of a regular pentagon in a given circle (IV, 10, 11) and the inscription of the regular icosahedron and dodecahedron in a given sphere (XIII, 16, 17). In all these instances, Euclid's treatments are in the synthetic manner. But Eutocius' text of Menaechmus' solution of the cube-duplication suggests through its adoption of the double mode that these other problems might also have owed their initial solutions to analytic treatments. Indeed, Pappus preserves a set of alternative constructions of the inscribed regular solids effected by means of analysis and synthesis.⁴⁴ Proclus relates that Plato taught the method of analysis to Leodamas for use in his geometric researches and that Eudoxus used analyses in his studies of the "section" (*tomē*).⁴⁵

It will be useful to reconstruct an analysis along the lines suggested by Euclid's synthesis of the problem of inscribing the regular pentagon in a given circle. As we shall later see, a comparable analysis is discernible as the basis of Archimedes' solution of the inscription of the regular heptagon (Chapter 5). Assume then that the regular pentagon $ABCDE$ has been inscribed in the circle [see Fig. 13(c)] and draw the diagonals AD , BE meeting in F . In the triangle AFF , the angles at A and E are equal, intercepting the equal arcs DE , AB , respectively. Since angle BAD intercepts the double arc BCD , it will be double the angle FAE . Since, furthermore, the angle BFA is external to triangle FAE , it will equal the sum of angles FAE , AEF , that is, twice angle FAE . Thus, angles BAF , BFA are equal, so that $AB = BF$. Since F is the vertex of the isosceles triangle FAE , it lies on FG , the perpendicular bisector of the given side AE . Its position may thus be given via *neusis*, in which the intercept FB from a line drawn from the



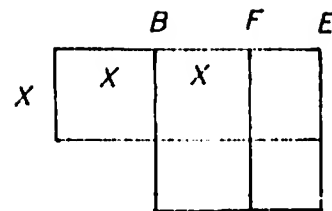
(c)

Figure 13

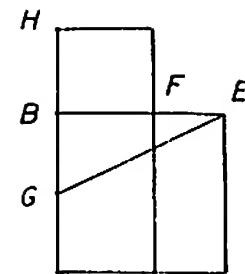
given point E shall have a given length ($= AB$) and lie between a given line (FG) and a given circle (that with center A and radius AB). The corresponding synthesis is evident, and in fact is nothing other than a variant of Hippocrates' third lunule construction.⁴⁶

Euclid's method in IV, 11 does not employ a *neusis*, however, so that the implied analysis must have followed a different course. If we adopt the same figure as before, we have that the triangles ABE, FAE are similar, since their base angles (at B, E, and A) intercept the equal arcs AE, AB, ED, respectively. Thus, $AB : BE = EF : AE$, and since $AB = AE = BF$, it follows that $BF^2 = EF \cdot EB$. We have thus reduced the problem of inscription to one of dividing a line into segments satisfying this second-order relation. The latter problem, familiar to the ancients as the division into "extreme and mean ratio" (cf. *Elements* VI, Def. 3), and to us as the "golden section," can be effected via the methods of the "application of areas," as Euclid does in II, 11 and again in VI, 30.⁴⁷ In the latter, the given area BE^2 is applied hyperbolically to the given line BE; that is, $BE^2 = X(BE + X)$ [see Fig. 13(d)]. Setting $BF = X$, it follows that $BF^2 = BE \cdot EF$, as required. But in II, 11 an independent method is used where BF is constructed as the difference of $\frac{1}{2} BE$ and the hypotenuse of the right triangle of legs BE, $\frac{1}{2} BE$ [see Fig. 13(e)].⁴⁸ The alternative method would seem to indicate an origin at an earlier stage of the development of these techniques.⁴⁹

If Euclid's procedure drew from the *neusis*-related analysis just given, we would expect a synthesis like this: divide the given line BE at F as specified in the analysis; complete the isosceles triangle with BE as base and legs BA, AE each equal to BF; circumscribe the circle about BAE; then BA, BE are the side and diagonal, respectively, of the regular pentagon inscribed in this circle. If we then inscribe in the given circle an angle equal to ABE, we shall have determined an arc A'E' subtended by a side of the regular pentagon inscribed



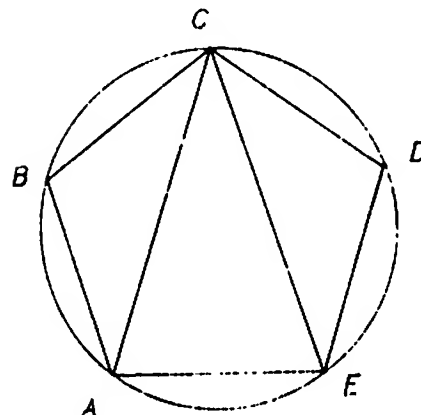
(d)



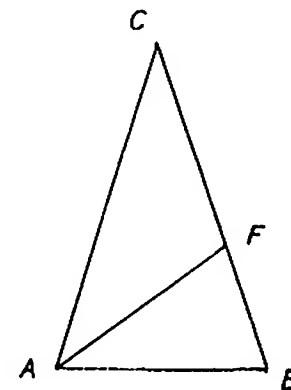
(e)

in it, from which the figure is readily completed. This approach is of course entirely within the scope of the early techniques; it is sufficiently straightforward in conception to make one uncomfortable in denying that the ancients knew it. But the synthesis given by Euclid in IV, 11 is quite different. We thus receive no encouragement from him in supposing the ancients followed this method,⁵⁰ and must frame another analysis as the basis of his method.

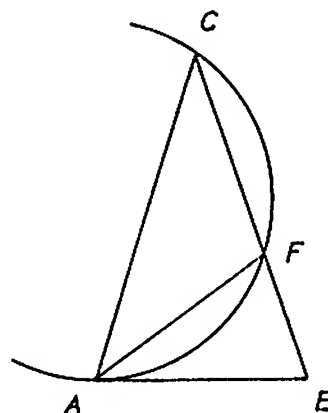
Assume again the regular pentagon ABCDE inscribed in the circle and now draw the diagonals AC, CE [see Fig. 13(f)]. It is evident that the triangle ACE is isosceles and that each of the angles at its base is twice the angle at its vertex. The inscription thus reduces to the construction of an isosceles triangle of this form; the synthesis of the latter problem is given by Euclid in IV, 10 in order to effect the inscription in IV, 11. To construct the triangle, we can produce the following analysis: assume ACE is isosceles, where the angle at A is twice that at C [see Fig. 13(g)]. Let AF bisect the angle at A, whence triangles AFE, CAE are similar. Thus, $EF : EA = EA : AC$, or $EA^2 = EF \cdot AC$. Now, $EA = FC$,



(f)



(g)



(h)

Figure 13

each being equal to line AF in the isosceles triangles AFE, FAC, respectively. Furthermore, $AC = EC$. Thus, $FC^2 = FE \cdot EC$. In this way, the construction of ACE reduces to the division of a given line CE in accordance with the extreme and mean ratio, as given in II, 11 (or VI, 30).

I have no doubt that the analysis underlying IV, 10—by which the construction of CAE is reduced to finding F such that $FC^2 = FE \cdot EC$ —took this form. The techniques of proportions and similar figures were already prominent in the ancient geometry quite early, as we have seen through the problem-solving efforts of Hippocrates. Moreover, treatments survive of closely related problems which adopt precisely this configuration of similar triangles.⁵¹ Nevertheless, the synthesis Euclid presents in IV, 10 does not merely retrace the steps of this analysis. Euclid assumes the division of CE such that $CF^2 = FE \cdot EC$ and completes the triangle CEA such that $EC = CA$ and $EA = FC$ [see Fig. 13(h)]; it is to be shown that the angles at A and E are each twice that at C. Draw AF and introduce the circle AFC. Since $EA^2 = FE \cdot EC$, EA is tangent to the circle (*Elements* III, 37), whence angles FAE and FCA are equal (III, 32). Euclid next proves that angles EFA and EAC are equal;⁵² since the triangle ECA is isosceles, the latter angle equals FEA, so that the triangle FEA is also isosceles, whence $EA = AF$. But by construction, $EA = CF$; angles FCA and FAC are thus equal base angles in the isosceles triangle FAC. The exterior angle EFA equals their sum, or twice the angle at C. Thus, in the triangle CEA, the base angle EAC (which equals angle EFA) is twice the angle at the vertex C.

Our reconstruction of the analysis has not prepared us for Euclid's introduction of the circle and the angles formed by its tangent. But these moves are quite naturally motivated by Euclid's pedagogical decision to defer the presentation of the techniques of proportion until later in his treatment of plane geometry (i.e., in Books V and VI). The constructions of the regular polygons in Book IV must thus be effected exclusively via methods of congruence. Thus, the

equality of angles EAF and ACE, which could have followed at once from the similarity of the corresponding triangles through the proportionality of their sides, must now be secured via the tangent EA to the circle AFC. This discrepancy raises a subtle point for the interpreter. Euclid's formalized organization of the *Elements* need not have followed the order of discovery; although it is conceivable that the whole body of plane geometry in Books I–IV was worked out prior to the adoption of proportion techniques by the ancient geometers, I find that only remotely plausible as a position on the historical development of the field. Since the purpose of an analysis of a problem is fundamentally heuristic, it is entirely feasible that this part utilize methods formally excluded at the stage of synthesis. For it is the synthesis alone which has formal status. In the case of the construction in IV, 10 at issue here, one could fashion an analysis which conforms step for step with Euclid's synthesis.⁵³ But that would be an exercise without either pedagogical or heuristic interest. In the light of this, Euclid's omission of analyses in his treatments of construction problems in the *Elements* follows naturally from the editorial decision to formalize the exposition of this material.

To the extent that the *Elements* do reflect the earlier procedures—certainly in more general respects, if not always on points of specific detail—we may submit the reconstructions above as a specimen of the problem-solving field among the 4th-century geometers. Characteristic of the method are the exploitation of analyses and the reduction of the desired configurations to conditions relating to the application of areas. From this period we may cite two further witnesses to the adoption of this approach: one is a passage from Plato's *Meno*, the other a set of remarks by Proclus on a geometer named Amphinomus.

The *Meno* (86 e – 87 h) introduces a method of reasoning “from hypothesis” which, but for its name, is identical to that of “reduction” as used, for instance, by Hippocrates in his attack on the cube duplication. The issue in the dialogue is to establish whether virtue is teachable. In the absence of a satisfactory definition of what virtue is, Socrates proposes that one consider a hypothesis from which to pursue the examination of the main question.

I say “from hypothesis” in the manner that the geometers often make inquiry, whenever someone has asked them, for instance about an area, whether this area here can be stretched out as a triangle in this circle here, one would say ‘I don’t yet know whether this is of such a sort, but I think that as a certain hypothesis the following will assist in the matter. If this area is such that the one who has stretched (it) along its given line (makes it) fall short by an area such as is the stretched (area) itself, then it seems to me that a certain result follows, but a different result, if it is impossible that these things be done. Having hypothesized, then, I wish to say to you whether the result about its stretching in the circle is impossible or not.’⁵⁴

Like all too many of the mathematical passages in the dialogues, this one leaves much to be desired in terms of clarity. Plato's phrasing here is quite free, whether by reason of his deliberate choice to suit the colloquial context of the dialogue, or perhaps through imprecision in the mathematical terminology of his day. For instance, the procedure of “stretching in” (*enteinein*) is clearly the inscription

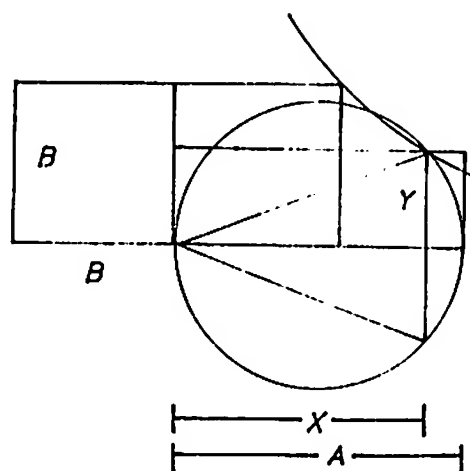


Figure 14

of one figure within another. But can one assume that the procedure of "stretching along" (*parateinein*) refers to the application (*paraballein*) of area, as would appear to be the case here? Is the area which falls short (*elleipein*) "by such as is" (*toioutōi, hoion ēi*) the applied area similar to it, or equal to it? Does "its given line" signify the diameter of the given circle, or perhaps some other line serving as a side for the figure of application?

These obscurities have occasioned dozens of efforts of interpretation. But one of the oldest attempts still remains the most plausible and widely accepted.⁵⁵ The problem as initially posed requires the inscription of a given area as a triangle in a given circle. We may figure the given area as a square of side B and denote by A the diameter of the given circle [see Fig. 14]. There is no loss of generality in requiring that the inscribed figure take the form of an *isosceles* triangle. In accordance with the mode of analysis, assume the figure has been inscribed as specified; let the altitude X of the triangle lie along the diameter, and let Y be half its base, set at right angles to the diameter. The condition on the area thus stipulates that $XY = B^2$. Furthermore, since Y is a half-chord in the circle, Y is the mean proportional between X and $A - X$, the two segments of the diameter it cuts off; that is, $X : Y = Y : A - X$. In this way the problem is reduced to another one, corresponding to that stated in the second part of Plato's passage: the given area B^2 is now to be applied to the given line A such that the area by which it falls short (namely, the rectangle of sides Y and $A - X$) be similar to the area applied (the rectangle of sides X and Y). Thus, the passage may be viewed as based on this reduction or analysis of the problem of inscription.

If we complete the analysis, in the manner suggested by Menaechmus' treatment of the cube duplication, we will observe that the condition on areas ($XY = B^2$) entails that X and Y are coordinates in a given curve, namely, the

hyperbola having as its asymptotes the diameter of the circle and the tangent drawn at its endpoint, where the hyperbola passes through the point whose coordinates both equal B . We have proposed that Menaechmus introduced such curves in the form of pointwise constructions corresponding to the defining area relation. But whatever construction he used, it would perfectly suit the purposes of the problem discussed in the *Meno*. This effectively details the principal objection raised by some against this interpretation of the passage: that it supposes a problem incapable of solution by the methods available to pre-Euclidean geometers.⁵⁶

A remarkable feature of the *Meno* passage is that it expresses the mathematical project not as the actual solution of a problem, but rather as the determination of the possibility of its solution. This has led many to view the passage as discussing a "diorism," that is, the statement of the necessary condition for the solvability of a problem. But in the mathematical literature diorisms have the form of explicit conditions on the givens of the problem. In the present case, this might be the statement that the given area must be less than the area of an equilateral triangle inscribed in the given circle; having verified this relation to hold for particular values of the givens, one would know that the problem is solvable in this case, even before one has begun the solution of the problem as such. Although it is often the case that the analysis of a problem reveals the appropriate form of its diorism, nevertheless, the articulation of the diorism is quite different from the analysis or reduction of the corresponding problem. We thus have to explain why Plato here frames this example of problem reduction as if it were equivalent to the determination of possibility.

According to the usual chronology of Plato's works, the *Meno* is transitional between the earlier and the middle dialogues, so that its composition is placed around 385 B.C.⁵⁷ This is two decades or more before the activity of Menaechmus, a disciple of Eudoxus whose association with the Academy falls in the period after 370 B.C. Some of the geometric techniques available to Menaechmus would not yet have been introduced at the earlier time of the *Meno*, and these might include the methods of curve drawing discussed above. Thus, Plato's emphasis on the possibility of the inscription problem might be taken to signify that geometers had then discovered the diorism, but not the actual solution of this problem. For a proof of the diorism can be effected via elementary methods of construction,⁵⁸ but this is not true of the inscription problem itself. With the introduction of the methods affiliated with Menaechmus' cube duplication, not only could the problem now be solved, but one also would obtain an alternative proof of the diorism; for the problem of application is possible only when the auxiliary hyperbola (in the derived problem) intersects the circle, and in the limiting case, where the curves meet at a single point of mutual tangency, one finds that $X = \frac{1}{4}A$, equivalent to the case of the equilateral triangle.⁵⁹

Supporting this view is one's intuition that the successful solution of the inscription problem could not long have antedated Menaechmus' solution of the cube duplication. But the form of the *Meno* passage must surely have been affected by certain dramatic and dialectical considerations at least as much as

by these historical and technical ones. Plato's principal concern in the dialogue is over a question of possibility: the teachability of virtue. It thus suits his purposes to frame the geometric example in like manner, as a determination of the possibility of a construction, even though the geometric method illustrated is that of reduction for the solution of problems. Further, Plato's account here is likely to reflect his own interpretation of what the true goal of such geometric activities is: not the finding of constructions, but rather the establishment of their possibility, that is, the demonstration of the existence of the solving entities. In the later mathematical writings problems are often asserted in the form, "it is possible (*dynaton estin*) to construct . . .," and taken literally, this might indicate an existential motivation underlying the activity of problem solving. I argue below (cf. also Ch. 8) that this provides a poor account of the objectives of the geometric research in this period. But geometers' actual motives need not much have affected Plato's view of them. I think that the special concerns of his general philosophy could well have encouraged Plato to represent, or misrepresent, the geometers' search for constructions as essentially an investigation into their possibility. This would account for the discrepancy between the *Meno* passage and the standard format in mathematical writings.

Proclus reports that Leon, a younger contemporary of Plato, "found diorisms," so that we may infer additional evidence of interest in such studies of problems (*In Euclidem*, pp. 66 f). Proclus elsewhere provides an example further revealing the use of techniques of area application in the analysis of problems.⁶⁰ A certain geometer named Amphinomus is said to have distinguished problems as "ordered," "intermediate," or "unordered" according to whether they permit of one, several, or an unlimited number of solutions. By way of illustrating this last kind, Proclus cites the problem of dividing a given line segment into three parts in continued proportion. If one divides the given line into segments in the ratio of 2 : 1 and one applies the square of the smaller segment to the larger segment as to fall short by a square, then the given line will be divided into three equal parts which are thus in continued proportion. If the given line is then divided into segments in any ratio greater than the double, say the triple or any larger multiple, and if in similar fashion the square on the smaller segment is applied to the larger as to fall short by a square, then again the line will have been divided into three parts in continued proportion. Let us designate by X the larger of the initial segments, by Y the smaller; then the procedure of application produces Z such that $Y^2 = Z(X - Z)$. It thus follows that the segments Z, Y, $X - Z$ are in continued proportion, while their sum equals the given line. Since the application is possible only when Z is less than or equal to one-half X, whence the maximal value for Y is seen to be one-half X, Proclus' initial choice of the division $X : Y = 2 : 1$ represents the limiting condition for the possibility of solving the stated problem. One would suppose that a "diorism" to this effect had been worked out in the original treatment. But as the number of larger multiples is unlimited, each yielding a solution, the problem has an infinity of solutions and so falls under Amphinomus' class of "irregular" (or "unordered," *ataktia*) problems.⁶¹

This is a nice specimen of the use of application techniques for problem-solving. A 4th-century provenance is indicated, but not entirely secured. One would suppose that Amphinomus himself cited this problem as an instance of his "unordered" class, although its appearance in this context might possibly be due to a later editor like Proclus. What we know of Amphinomus otherwise is contained in three additional references by Proclus, all of which associate him with 4th-century figures. Amphinomus is said to have upheld the view of Aristotle, that geometry does not inquire into the causes of things (*In Eucl.*, p. 202).⁶² He is linked with Speusippus, Plato's nephew and successor, as maintaining that all propositions in geometry should be framed as theorems, for this form best captures the eternal and nongenerable nature of geometric entities; by contrast, Proclus continues, the mathematicians allied with Menaechmus emphasized the problematic element in geometric research, whether this seeks the production of an entity or the determination of its properties (*In Eucl.*, pp. 77 f).⁶³ Furthermore, when Proclus observes that not every proposition has a true converse, but that a proposition will be convertible when it refers to the primary or essential attribute of a class, he adds that "these things did not escape the notice of the mathematicians around Menaechmus and Amphinomus" (*In Eucl.*, p. 254). This passage recalls a discussion by Aristotle, to the effect that a proper definition should pick out attributes which apply to all and only the members of the class defined (*Posterior Analytics* I, 4, 5). The special interest which geometers might have in the convertibility of propositions is clear from another comment by Aristotle:

If it were possible to prove something true from something false, the procedure of analysis (*analyein*) would be easy; for it would convert of necessity. For let us posit that A obtains; from this supposition these things follow which I know do obtain, namely B. Then from these things I can show that the former (A) obtains. Now the things in mathematics tend to be convertible, for they posit nothing contingent, but only definitions—and in this they differ from the things in dialectical philosophy (*ibid.*, I, 12).⁶⁴

Aristotle here portrays analysis as a deductive procedure leading from the desired premise (or construction), hypothesized to be true (or constructible), to another one known to be true (or constructible). This accords with the examples we have already discussed, as well as with those in the later mathematical literature. Moreover, Aristotle's awareness of the relevance of the issue of convertibility reveals that current mathematical practice had recognized the need for the corresponding synthesis to effect the demonstration; where steps in the analysis are not simply convertible, one would have to identify the conditions under which conversion is valid, that is, the conditions of the diorism. Aristotle thus indirectly attests to the maturity of this method among the geometers of his generation.⁶⁵

We thus see that Menaechmus' method for solving the cube duplication typifies an important field of the pre-Euclidean geometry. It is but one of several known instances employing the technique of application of areas, a versatile instrument for the analysis of problems. Indeed, the field of analysis had been

sufficiently advanced by around the middle of the 4th century B.C. that certain special features of the logical structure of analysis began to attract the attention of philosophers. For instance, the "diorism" expressing the conditions for the possibility of solution results from consideration of the conditions permitting the steps in the analysis to be converted, so to produce the formal synthesis. Among some there already appears to be emerging a self-conscious adherence to the formalist ideal. In the question as to whether one ought to frame geometric propositions as theorems or as problems, Proclus finds a spokesman for the theorematist view, as one might predict, in the person of a 4th-century Platonist philosopher. But he draws from the geometer Menaechmus a sense of the priority of problems. This division is surely ominous. For a preoccupation with the formal exposition of theorems, at the expense of the analysis of problems, could hardly avoid retarding the effort to discover new solutions. On the other hand, the formal work of synthesis can sometimes raise difficulties demanding geometric insight for their resolution. The efforts of Eudoxus, to which we now turn, illustrate this impressively.

EFFORTS TOWARD THE QUADRATURE OF THE CIRCLE

As geometers sought ways actually to construct the square equal to a given circle, they came to recognize that the underlying theorems on circle measurement posed difficulties of their own. Hippocrates' measurement of the lunules, for instance, must assume that circles have the ratio of the squares on their diameters. Any proof which Hippocrates could have offered for this theorem, such as the one we proposed above on the model of Antiphon's sophism on the circle quadrature, must inevitably have adopted a naive manner of limits, for the formally correct method, as presented in Euclid's Book XII, owed its first introduction to Eudoxus.

Aristotle names the 4th-century Sophist Bryson, along with Hippocrates and Antiphon, as having attacked the problem of circle quadrature.⁶⁶ But the later commentators Alexander (3rd century A.D.) and Proclus, Ammonius, and Philoponus (5th and early 6th centuries), who inform us of the details of Bryson's argument, seem to know only this much⁶⁷: having circumscribed a *square* about the circle and then inscribed another within it, and having taken a third *square* intermediate between the first two, Bryson claimed that this *square* equalled the circle, "for things greater than and less than the same things are equal."⁶⁸ On the meaning and criticism of this argument, the commentators disagree. The simplistic reading leads at once to absurdity: one might claim on the same basis that 9 and 10 are equal, since they are both greater than 8 and less than 12.⁶⁹ A more sophisticated interpretation takes into consideration the sequence of *all* circumscribing and inscribed regular polygons: then "the rectilinear figure" which is less than all the circumscribing polygons and greater than all the inscribed polygons will equal the circle, since the circle satisfies the same inequalities.

Philoponus dismissed this latter effort by Proclus to refurbish Bryson's principle in the form "to that than which there is a greater and a lesser there is also

an equal."⁷⁰ It is hard to see the force of Philoponus' attempted counterexample, however. To be sure, the "horn angle" contained between a circle and its tangent is less than every rectilinear acute angle, while the corresponding mixed angle between the circle and one of its diameters is greater than every rectilinear acute angle. But when he observes that this has not established the existence of an equal angle,⁷¹ we might well ask what angle he is referring to. Through further elaboration of this class of mixed angles, Philoponus shows that he would deny what the modern theory of curves would affirm: that the angle at which two curves intersect each other is taken equal to the angle between their tangents at the point of intersection. Despite the fact that these angles are obviously *different* from each other geometrically, it is important to accept them as quantitatively *equal* if quantitative statements about tangents are to be possible at all. It is interesting that writers in the ancient tradition of geometric optics adopt this principle quite freely, manifesting none of the qualms expressed here by Philoponus.⁷²

Philoponus further objects to Proclus' interpretation of Bryson on the grounds that it misses the point of the problem of circle quadrature:

Those squaring the circle did not seek (to establish) whether it is possible that there exists a square equal to the circle, but rather thinking that it can be so, they tried to produce a square equal to the circle.⁷³

But in accepting Proclus' argument as an existence proof, he is perhaps too kind. For Proclus has far too easily slipped in the "rectilinear figure" presumed at once greater than all the inscribed figures and less than all the circumscribed ones. If, for instance, this intermediate figure is being generated along with the sequence of bounding polygons, then it will effectively turn out to be an infinite-sided regular polygon, namely the circle itself. But a subtle modification of the argument can avoid this difficulty. For each bounding polygon, let us construct the equal square. As the number of sides increases, the sequence of inscribed polygons gives rise to a sequence of squares of increasing area, while the sequence of circumscribing polygons gives rise to a sequence of squares of decreasing area. Since, furthermore, all the squares in the one sequence are less than all the squares in the other, there will be a square intermediate between the two sequences. That the difference between the circumscribed and the inscribed polygons can be made arbitrarily small is geometrically obvious and can be secured by a relatively straightforward bisection argument.⁷⁴ Hence, the intermediate square is unique, from which its equality with the circle follows from Bryson's principle. In this way, a Weierstrassian procedure for *defining* the magnitude of the circle might be foreshadowed, if dimly, in Bryson's approach. This is the interpretation adopted by some modern writers.⁷⁵ But we do well to note that the ancient commentators on Bryson seem to have no firm account of the details of his argument and do not themselves give a precise statement of his principle and a valid form of its application, despite their own acquaintance with the limiting methods in the tradition after Eudoxus. Indeed, the issue at the heart of Bryson's sophism, the articulation of the conditions for equality between

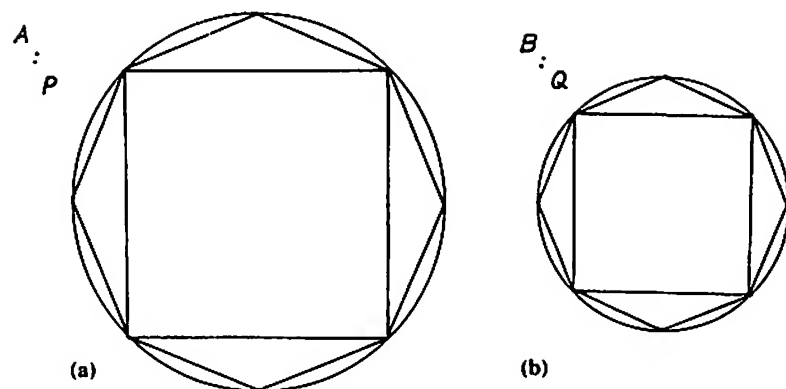


Figure 15

continuous magnitudes, receives no fully satisfactory explication either among the philosophers or among the geometers of antiquity. One encounters instance upon instance in the writings of Euclid, Archimedes, and others where notions of continuity enter, based on intuitive assumptions rather than on explicit formulations.⁷⁶

Bryson's argument must reflect an interest among mid-4th-century geometers in the formal questions entailed by the assumption that problems like the circle quadrature do indeed have solutions. In this context Eudoxus devised a special technique, called by modern writers the "method of exhaustion" or the "indirect method of limits," which could secure the demonstration of theorems like those required by Hippocrates, even if it left in abeyance some of the deeper foundational issues implicit in Bryson's argument. The Eudoxean method, as we learn principally from Archimedes, is the basis of the measurements of the circle, pyramid, cone, and sphere in the manner now extant in Euclid's Book XII. Archimedes himself, whom we may perceive to be working from Eudoxus' treatments in preparing such early works of his own as the *Dimension of the Circle*, applies essentially the same technique one finds in the *Elements*.⁷⁷

Let us then sketch Euclid's theorem on the area of the circle (*Elements* XII, 2) as a representative instance of this method. One proposes to demonstrate that circles have the ratio of the squares on their diameters. Let the circles be A, B, and the squares be C, D [see Fig. 15]. If it is not true that $A : B = C : D$, then for X such that $A : X = C : D$, X will either be less than or greater than B. Assume first that X is less than B. We now inscribe in B a regular polygon, form from this the inscribed regular polygon having twice as many sides, and continually repeat this procedure on the newly formed polygons. Since the difference between the circle and the polygon is reduced by more than half at each step in this procedure,⁷⁸ we will eventually obtain a polygon Q such that $B - Q$ is less than $B - X$ ⁷⁹; that is, Q is greater than X. Now inscribe in A the regular

polygon P similar to Q. Since $P : Q = C : D = A : X$ and P is less than A, it follows that Q is less than X. This contradicts our having found Q greater than X. Hence, X cannot be less than B. One might conceive an exactly analogous procedure utilizing circumscribed polygons whereby the alternative assumption that X is greater than B is reduced to a contradiction. But Euclid perceives a much simpler alternative. Using the same notation as before, we posit the quantity Y such that $Y : B = A : X = C : D$. Since X is greater than B, Y is less than A, so that we are in precisely the same situation as before: the existence of such Y leads to a contradiction, so that X cannot be greater than B either. Hence X equals B, and the stated proportion of the circles and the squares of the diameters is thus affirmed.⁸⁰

The procedure of inscribing regular polygons and successively doubling the number of sides has an evident precedent in Antiphon's circle quadrature, although Eudoxus surely drew it from Hippocrates' proof of the theorem on the ratios of circles,⁸¹ for Eudoxus' proof had the effect of rigorizing the treatment of this very same theorem. The key insight in the revised version is the appeal to successive bisection in the context of an indirect proof. The lemma that any finite magnitude can be reduced by repeated bisection so as to become less than any assigned finite magnitude of the same kind is characteristic of all the limiting arguments in *Elements* XII and is retained in the majority of limiting theorems of Archimedes. It is proved in *Elements* X, 1; but the method employed there does not conform to the style in Book XII.⁸² One detects that this lemma could be deemed sufficiently obvious as not to require an explicit proof at the initial stages of Eudoxus' theory, but that later disciples sought to ground it on a more fundamental axiom, comparable to the "Archimedean axiom" on continuous magnitudes.

It is worth noting certain features which do *not* characterize Eudoxus' method. Convergence is always one-sided, usually via a sequence of inscribed figures. The device of two-sided convergence, via sequences of paired inscribed and circumscribed figures approaching each other arbitrarily closely and hence approaching a common limit, is typical of Archimedes' more advanced measurements, but appears to have no precedent in earlier technical writings.⁸³ It thus seems implausible that Bryson's sophism on circle quadrature included a clear representation of the two-sided manner of limits, so that the relatively unsophisticated form of the argument suggested by the report of Alexander (where the inscribed rectilinear figure continuously increases toward the circumscribed figure, and so at some time passes through a figure equal to the circle in magnitude) seems more likely to capture Bryson's attempt than the more elaborate version proposed by Proclus. Another striking feature of Eudoxus' method is its failure to utilize the forms of the theory of proportion given in *Elements* V. For comparison, we may reconstruct an alternative treatment in the Euclidean manner⁸⁴:

To prove $A : B = C : D$, let us suppose the contrary. Then, first, let A : B be greater than C : D. There exist integers m, n such that $mA > nB$, while $mC \leq nD$.⁸⁵ Since $A > n/mB$, we may inscribe in A a regular polygon P such that $P > n/mB$.⁸⁶ Let Q be the polygon similar to P inscribed in B. It follows that $mP >$

nQ . Now P and Q are similar, so that $P : Q = C : D$; since $mC \leq nD$, one has $mP \leq nQ$. This contradicts the previous inequality relating P and Q . If, then, $A : B$ is less than $C : D$, the inverted ratios satisfy the opposite inequality, so that the same contradiction results. Thus, $A : B = C : D$.

The most straightforward reason for Euclid's failure to adopt such a procedure in the case of this and comparable theorems in Book XII is that Book XII owed its origin to earlier work than that which yielded the proportion theory presented in Book V. While the latter is usually ascribed to Eudoxus, an alternative proportion theory conforming quite closely to the style adopted in Book XII is implicit in certain proofs in Archimedes and other later writers.⁸⁷ One would thus infer that Book V reflects modifications in Eudoxus' theory by his followers, but that Euclid's treatment of the limiting theorems in Book XII adheres to the methods actually used by Eudoxus.

The principal advantage which Euclid could have obtained by employing this reconstructed alternative method for the theorem on circles is its elimination of the need to hypothesize the existence of the magnitudes X and Y satisfying the proportionalities $A : X = C : D = Y : B$. This hypothesis of the existence of the fourth proportional is a fixture in all applications of the Eudoxean-Archimedean method of limits, as well as in the reconstructed Eudoxean proportion theory mentioned above. It is one of several striking instances of the appeal to nonconstructive assumptions within the ancient geometry.⁸⁸ Like Bryson's intermediate square, these magnitudes can be assumed to exist via appeal to some intuition of the nature of continuous magnitude in general. Here one perceives no concern whatever to ground such appeals on explicit constructions of the hypothesized terms. Indeed, Philoponus' remark on the circle quadrature entirely separates the issue of the *existence* of the square equal to the circle from that of the *construction* of that square, in that the former is assumed from the start as the basis for the latter. Ever since Zeuthen, the view that constructions serve the purpose of existence proofs has been a favorite interpretation among writers on ancient geometry.⁸⁹ The present counterinstances should give one pause in subscribing to this view as if it were one the ancients would have held of their own efforts.

Among the ancient constructions of the circle quadrature, the method utilizing the special curve called the "quadratrix" is often assigned a pre-Euclidean origin. Pappus cites its use by Menaechmus' brother Dinostratus and by Nicomedes (late 3rd century B.C.), and the latter is affirmed by Iamblichus.⁹⁰ But in two places Proclus associates this curve with a geometer named Hippias.⁹¹ If this is indeed Hippias of Elis, the Sophist, a contemporary of Socrates, the quadratrix would be the earliest known instance of a special curve among the ancients, and Hippias would deserve to be ranked foremost in mathematical expertise not only among the Sophists of his generation, but among the geometers as well.⁹² But the technical sophistication of this would then so strongly foreshadow discoveries by Eudoxus and Archimedes that one is immediately put on guard. A too hasty acceptance of its attribution to such an early figure threatens to distort one's

natural view of the chronology of the ancient technical methods by as much as a century or two.

Left to a vote count among modern authorities, the attribution would win acceptance comfortably. Tannery and Björnbo defended the case quite insistently, and it has been adopted by Cantor, Heiberg, Heath, and many others since.⁹³ Dissident voices include Allman and Hankel among the older writers and Guthrie more recently.⁹⁴ The case rests entirely on the fact that Proclus does mention Hippias of Elis in one other passage, as an authority to support the claim that "Mamercus, the brother of the poet Stesichorus . . . had a reputation in geometry" in the time before Pythagoras.⁹⁵ Clearly, the context has nothing to do with the quadratrix and signifies nothing of the mathematical expertise of Hippias himself. Does this single passage then justify shifting the burden of proof onto those who doubt the identification of Hippias the sophist with Hippias the geometer? The substantial body of fragments from the Sophist give no evidence of interest or competence in technical matters.⁹⁶ To be sure, Plato has Socrates, in the dialogue *Hippias Minor*, address him as "a skillful calculator and arithmetician . . . the wisest and ablest of men in these matters," to which Hippias assents quite unequivocally; he is later noted as adept in geometry and astronomy as well, and indeed boasts of his own skill in all the arts, including the making of rings, seals, the very shoes, cloak, and tunic he was then wearing, not to omit the arts of memory, of writing prose and poetry of all forms, of music and so on.⁹⁷ The point is surely clear: Hippias is being ribbed as a self-styled jack of all trades, but an actual master of none in particular. This is slippery ground to base a claim of his accomplishments in the field of geometry. Can one so readily disregard the negative testimony of Aristotle, who cites the circle quadratures by Hippocrates, Antiphon, and Bryson, but none by Hippias? The later Aristotelians, for all their ample commentary on these passages on the circle quadrature, likewise omit any mention of Hippias. Yet we may infer from Proclus that this man's contribution was hardly a casual one:

Apollonius proved the properties of each of the conic lines, and Nicomedes (did the same) for the conchoids, and Hippias for the quadratrices, and Perseus for the spirics [i.e., the sections of the torus].⁹⁸

Thus named in the company of late 3rd-century figures like Apollonius and Nicomedes, known for entire treatises devoted to the properties of the curves associated with them, Hippias would appear to be credited here for a comparable treatment of a *class* of curves, the "quadratrices" (note the plural). Yet several writers, including Proclus himself, assign a share in the study of this curve to Nicomedes also; indeed, it was Nicomedes, according to Iamblichus, who appears to be responsible for naming the curve "quadratrix" (*tetragōnizousa*) in recognition of its role in the quadrature of the circle. One wonders what was left for Nicomedes to do, however, if Hippias the Sophist had so advanced the study of this curve more than two centuries earlier. When we consider further that the name Hippias was quite common,⁹⁹ how can we deny that the Hippias

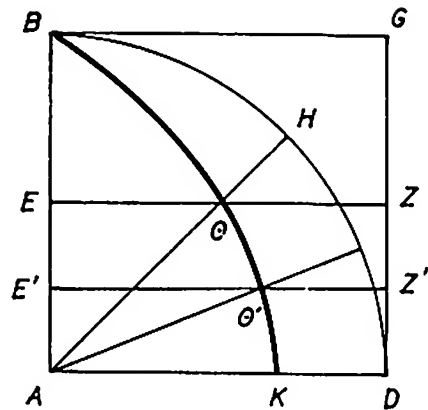


Figure 16

of the quadratrix was not the 5th-century sophist, but rather a 3rd- or 2nd-century geometer extending the work of Nicomedes?

Thus, strictly on the basis of textual evidence, the case for assigning this curve to the sophist is exposed as implausible. But if the technical implications of this attribution had been considered by its advocates, it would never have been entertained seriously in the first place. We shall now find it useful to provide a brief account of this curve, in order to show how it relates to 3rd-century elaborations of the Eudoxean methods.¹⁰⁰ With reference to the square ABGD, let us conceive that the ray AB rotates on A from an initial position at AB to a terminal position at AD, while in the same time BG moves parallel to itself from BG to AD (see Fig. 16). The moving lines will thus intersect at a point θ which traces a curve running from B to a point K lying on AD. One may show that the given length AB is the mean proportional between the terminal length AK and the length of the arc of the quadrant BD. Since one can construct the line which is the third proportional of the lines AK, AB,¹⁰¹ this line will equal the quadrant BD. Having thus rectified the arc of the quadrant, one can effect its quadrature, since its area is equal to one-half the product of its radius AB and its arc BD.¹⁰²

As one can well anticipate, the full proof of this construction, such as one finds it in Pappus, depends heavily on the use of limiting methods of the Eudoxean type.¹⁰³ It is set out in the indirect form and assumes properties of similar arcs whose formal proofs may be associated with Archimedes.¹⁰⁴ A certain subtlety of method is required even to secure the conception of the terminal position at K; for at this position the defining rays coincide. This need for an appeal to continuity to specify K occasioned misgivings over the method among later commentators like Sporus.¹⁰⁵ One can hardly imagine that it would have failed to spark discussion among 4th-century geometers and philosophers and that some

reflection of this would emerge in the comments by Aristotle and his interpreters. One further notes that the method actually effects a *rectification* of the arc BD. Yet Aristotle is quite ignorant of any such method, when in the *Physics* (VII, 4) he criticizes a more primitive method of the following sort:

What then is the situation (with respect to motions) in the case of the circle and the straight line? For it would be absurd to suppose that this cannot be moved in a straight line similarly to that in a circle, but that the straight must necessarily move either faster or slower Nor does it matter at all if one were to say that the straight does in fact necessarily move faster or slower. For then the arc shall be greater and less than the straight line, so that also (it shall be) equal. For if in the time A the one traversed the (line) B, the other the (arc) G, then B would be greater than G. For that is how "faster" was defined. Then also the faster (shall traverse) the equal in a lesser (time), so that there shall be some part of (time) A in which that (moving along) B shall traverse the equal of the circle, while that (moving along) G (shall traverse) G in the whole time A. To be sure, then, if these are compatible (*sympblēta*),¹⁰⁶ what has just been said will follow, namely, a straight line shall be equal to a circle. But they are not compatible; hence, neither are the motions.¹⁰⁷

At the heart of this argument is the Brysonian principle: that if a varying magnitude can become now greater, now less than a specified value, at some time in the interval it will equal that value. Here, the inequalities are arranged through the assumption that circular and rectilinear motions can be compared to each other via the relational terms "faster" and "slower." It is hardly an advertisement of Aristotle's geometric insight that he dismisses so perfunctorily this perfectly plausible rationale for the existence of the line equal to the circle. But one may note that Aristotle invariably treats with great respect the technical findings of such colleagues as Eudoxus and his followers.¹⁰⁸ He would hardly have set aside so casually the present argument, were its conclusion and the kinematic conceptions supporting it already secured through the quadratrix construction by Dinostratus, let alone by Hippias over a half-century earlier.

The lemma associating the rectification of a circular arc with the area of its sector is indispensable if one is to derive from the quadratrix a solution of the circle quadrature. A proof is extant as the first proposition of Archimedes' *Dimension of the Circle*, and Pappus cites Archimedes explicitly both in this passage on the quadratrix and in numerous other places in the *Collection* where this same lemma is introduced. Other writers like Hero, Theon, Proclus, and Eutocius invariably assign this result to Archimedes, often with specific reference to the *Dimension of the Circle*.¹⁰⁹ We have no evidence whatsoever to justify assuming an awareness of this result by an earlier Greek geometer. This must certainly rule out any idea of its use by Hippias a century and a half before Archimedes. But it also discourages assigning its use by Dinostratus, around three-quarters of a century before Archimedes, for the manner of proof adopted by Archimedes involves only such techniques as were available to any geometer in the Eudoxean tradition.¹¹⁰ For instance, he uses the familiar bisection method in the context of two independent applications of one-sided convergence, first

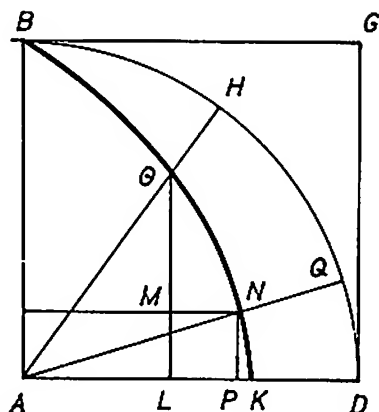


Figure 17

by inscribed regular polygons, then by circumscribed polygons. None of the more sophisticated limiting methods, such as appear in his *Sphere and Cylinder*, are brought to bear on the circle measurement. Can one comfortably accept the implication that pre-Euclidean geometers freely admitted this lemma in advanced constructions, like that involving the quadratrix, but that its elementary proof was overlooked for decades until Archimedes?

These considerations thus reveal the utter implausibility of assigning the use of the quadratrix to a Presocratic like Hippias of Elis. But they raise the question of reconciling a post-Archimedean dating with the testimony of Pappus that "for the quadrature of the circle a certain curve has been received by Dinostratus and Nicomedes and some other more recent figures; it takes its name from the property relating to it, for it is called by them 'quadratrix'."¹¹ As we have already noted, Iamblichus associated this naming of the curve with Nicomedes alone, but assigns no involvement on the part of Dinostratus. This suggests that Dinostratus might have introduced this curve in the context of a problem other than the circle quadrature. Pappus shows in detail how the quadratrix may be used for the division of an angle.¹² Let us propose to divide a given angle into three equal parts. We introduce the quadratrix BθK and draw the ray AH making with AD an angle equal to the given (see Fig. 17). Let AH meet the curve at θ and draw the ordinate θL; next divide θL at M such that ML is its third part. If a perpendicular to θL is drawn through M, it will meet the curve at a point, say N. Then the ray AN makes with AD an angle one-third that of the given angle HAD. This is evident at once from the motions which define the curve; for they entail that the ordinates have the same ratio as the associated arcs, e.g., that $\theta L : NP = \text{arc } HD : \text{arc } QD$, hence, also, as the associated angles (HAD, QAD).

It is not difficult to imagine how a curve of the above sort might be introduced for the purposes of the angle division. Aristotle's critique of the attempted

rectification of the arc suggests an extremely simple practical procedure: figure the given angle as HAD in a circle, and with a cord or flexible ruler lay out the arc HD; next straighten the ruler and divide the segment HD in the desired ratio (say, $QD : DH = 1 : 3$); now lay the ruler back against the circle to find the arc QD, whence angle QAD will be the desired part of HAD. Thus, the angle division brings forward a natural association of curvilinear and rectilinear magnitudes. Further consideration might lead one to try to correlate circular and linear motions, the one traversing arc BD uniformly in the same time that the other traverses the straight line equal to the arc BD, and then to replace the latter with the precisely similar motion traversing the straight line BA. Although it would hardly be a trivial step to contrive in this way a curve answering these motions and work out its application for the angle division, the way has been well prepared through the kinematic constructions of curves by Archytas and Eudoxus. In this way Dinostratus was in the position to convert the simple procedure, too hastily dismissed by Aristotle, into an ingenious method based on a new curve. A slightly different orientation of the same motions yields the spiral studied somewhat later by Archimedes and his colleague Conon.¹³ One would thus suppose that their discoveries relating to the spiral led Nicomedes to discover the use of the former curve for the circle quadrature, and so to dub it the "quadratrix." Only after him, around the beginning of the 2nd century B.C. at the earliest, could the geometer Hippias compile these findings into a writing on this curve.

This view of the origins and study of the quadratrix charges Pappus with an erroneous, or at least misleading, statement in his giving to Dinostratus a share in the use of this curve for circle quadrature. But he is hardly free of similar inaccuracies elsewhere. For instance, he writes thus of the early efforts on angle trisection¹⁴:

Having set out to divide the given rectilinear angle into three equal parts, the ancient geometers were at a loss for the following reason. We say that there are three kinds of problems in geometry: some are called planar, some solid, and some linear. [Pappus now explicates this classification in detail.] Since there is this difference among problems, the former geometers were not able to discover (a solution for) the cited problem of the angle, being by nature of the solid class, when they sought it by means of planes.¹⁵ For the sections of the cone were not yet familiar to them, and so they were at a loss. Later, however, they trisected the angle by means of the conics, using for its discovery the following *neuses*.

Pappus' thinking here is apparent. The angle trisection is assuredly a solid problem, that is, one solvable by means of conic sections. Hence, its *proper* solution would elude geometers until they had access to the requisite body of theorems on conics. The method he goes on to discuss betrays strong signs of a provenance near the time of Apollonius, however, so that Pappus might be construed to mean that no earlier methods of angle trisection had been worked out. This, of course, is an inference which no one would accept. Methods using the "quadratrix," the spirals or the *neuses* of Archimedes, or the conchoids of Nicomedes would all be assignable to periods earlier than the end of the 3rd

century. Pappus has allowed his use of the conceptions and terminology of the later development of geometry to distort his characterization of the earlier efforts, much as we have argued Eratosthenes did in alluding to the "cone sectioning" of Menaechmus. But if indeed Pappus supposed that the mere introduction of a certain curve, as by Dinostratus, carried with it a knowledge of the whole range of its applications, embracing both the solution of the angle trisection and that of the circle quadrature, we need hardly imitate his mistake. For it is surely plain that the latter effort at least depended on advances in geometry due to Archimedes.

GEOMETRY AND PHILOSOPHY IN THE 4TH CENTURY

The technical methods of geometry experienced significant advances at the hands of the group of geometers affiliated with Plato's Academy. Problems like the cube duplication and the angle trisection were attacked and solved in a variety of ways. Special curves were devised for these ends via ingenious intuitions of mechanical motions. Archytas, for instance, used the curve formed as the intersection of a torus and a cylinder for doubling the cube; of the same kind was the "curved line" of Eudoxus, arguably the ophiuride generated by the moving lines in the pseudo-Platonic solution; so also was Dinostratus' quadratrix, conceived for trisecting the angle. At the same time the techniques of the application of areas were being implemented for the solution of a range of problems, including the construction of regular polygons and solids. Menaechmus perceived how these give rise to second-order curves by which the cube duplication can be solved, although his method for producing the parabolas and hyperbola so defined is more likely to have been a pointwise procedure than the conception of sectioning the cone familiar to Euclid and his contemporaries several decades later. Furthermore, the method of analysis received considerable development in this period, exploited not only by geometers like Leodamas, Leon, Eudoxus, and Menaechmus, but even by a philosopher like Aristotle in his discussions of problem solving in general.

Responding to the philosophical environment of their efforts, these geometers markedly improved the formal basis of the organization and proof of geometric theorems. Proclus names about a dozen figures, most having an association with the Academy, who contributed to the composition of *Elements* of geometry, precursors of the various parts of Euclid's treatise.¹¹⁶ Particularly noteworthy were the insights by Eudoxus, who instituted techniques of limits and of proportions in a manner exemplary for its formal precision. The former filled gaps in Hippocrates' treatment of the measurement of the circle, showing how his use of infinite limiting procedures could be effected through finite sequences in the context of indirect proofs. Eudoxus' theory of proportions permitted the extension of theorems on proportions to the cases of incommensurable magnitudes, and so secured the technical advances in the study of irrationals, most notably by Theaetetus in the preceding generation.¹¹⁷ One thus sees that Eudoxus, however imbued with the spirit of formalism encouraged by his philosopher

colleagues in the Academy, nevertheless drew the principal inspiration for his technical efforts from the earlier geometers like Hippocrates, Archytas, and Theaetetus.

This convergence of geometric and philosophical interests has sorely tempted many scholars engaged in the study of these ancient efforts. Accordingly, no other period of the ancient geometry has been so thoroughly laced with inappropriate interpretive positions as this pre-Euclidean period.¹¹⁸ Concerning the early circle quadratures, one commonly attributes to Sophists like Antiphon and Bryson pioneering insights into the nature of limiting methods, while to Hippocrates one imputes blatant oversights and elementary logical errors.¹¹⁹ Democritus emerges as a precursor of Archimedes in the devising of a geometric method of indivisibles,¹²⁰ while Hippias of Elis is alleged to anticipate his discoveries on the measurement of the circle through the use of the quadratrix. Eudoxus is motivated not by the ambition to advance the technical achievements of earlier geometers, but rather by the concern to accommodate the arguments of Zeno and Aristotle against the use of infinities.¹²¹ Indeed, dialectical motives like the search for existence proofs are read into the whole ancient activity of solving geometric problems of construction.

Whence this tendency to distort the interpretation of the ancient geometry? Reconstructing the motives of others, even of near contemporaries, is always an uncertain enterprise. But I suspect, in this instance, that the particular intellectual background of the principal commentators is a significant factor. Tannery and Björmbom, for instance, those most responsible for the wide acceptance of the case relating to Hippias of Elis, are remembered primarily for their competence in philosophy and philology. How natural that they should seize upon the notion that a man of their own mind, like this Sophist, could have influenced the development of geometry! By contrast, no such presumption marked the view of Hankel, by profession a mathematician. One observes, furthermore, that the period of the late 19th and early 20th centuries witnessed a phenomenal shift in perspective on the nature of mathematics.¹²² Developments in analysis, geometry, number theory, and the theory of sets, for instance, revealed that certain formal issues, long ignored or unperceived, lay at the heart of important technical difficulties, and one might even maintain that the whole of mathematics was subsumed within logic. Hard on the heels of the satisfactory resolution of foundational questions in analysis and geometry, however, came a new "foundations crisis" in the field of logic itself. In all this, the close interaction of mathematical and philosophical inquiries was evident as a stimulating area for research. From this vantage point one could readily project a comparable form of interaction between the philosophical paradoxes raised by Parmenides and Zeno and the geometry of the early Pythagoreans; or between the philosophical repercussions of the discovery of irrationals and the choices of technical methods by the 5th- and 4th-century geometers. One hypothesized a paralysis of research, a renunciation of the use of proportions in geometry, and so on, until this impasse was finally broken through the efforts of Eudoxus.¹²³ One seemed not to care that the extant evidence, fragmentary as it is, indicates no signs of such paralysis or

renunciation, but rather a remarkable degree of continuity in the development of technical methods.

The alternative view advocated here acknowledges the significant interaction between geometry and philosophy over the special field dealing with the formal nature of proof, but that, on the whole, the two disciplines developed autonomously. Is that remarkably different from the course of modern mathematics? Save for those actively engaged in the study of foundational questions, how many mathematicians accept the general implications of those studies as an immediate factor in their own researches? It is interesting to consider the case of O. Becker, a prominent interpreter of pre-Euclidean geometry. As an adherent to the Husserlian phenomenology in the 1930's, Becker was especially sensitive to the questions of method raised by the earlier mathematical intuitionists. He thus probed the work of Eudoxus for signs of a similar concern.¹²⁴ If he did indeed discern formal aspects of the development of proportion theory never so clearly perceived by scholars before him, he nevertheless failed to find evidence of other concerns of the modern philosophy of mathematics, like the stricture against nonconstructive assumptions in proofs. But how many of Becker's own contemporaries shared these concerns in their researches?

This habit of reading present views into the past is hardly a recent phenomenon. The later Pythagoreans told the tale of one Hippasus doomed to shipwreck for his sacrilege of having divulged the secret of the irrational to the uninitiated.¹²⁵ The story of the Delian oracle seeks to link Plato directly with the early researches on cube duplication, and Eratosthenes converts it into a parable of Plato's philosophy of mathematics.¹²⁶ Here is material for theories of "foundations crises" and philosophers' influence on the progress of geometry, should anyone seek it. Let the ancients have their fun, spinning yarns to illustrate their favorite notions in philosophy. But we need not take these too seriously as guidelines for tracing the history of the ancient technical researches.

NOTES TO CHAPTER 3

¹ In *Euclidem*, p. 66.

² On these geometers, see my *Evolution of the Euclidean Elements*, Ch. 3, 7, 8. Short accounts may be found in the *Dictionary of Scientific Biography*.

³ Perhaps the most ambitious attempts to assess the mathematical component of Aristotle's thought are T. Heath, *Mathematics in Aristotle* and H. J. Waschkies, *Von Eudoxos zu Aristoteles*.

⁴ Cited by Eutocius in Archimedes, *Opera*, ed. Heiberg (2nd ed.), III, p. 96.

⁵ *Ibid.*, pp. 84–88. An extremely lucid account is given by B. L. van der Waerden, *Science Awakening*, pp. 150 f. See also O. Becker, *Mathematisches Denken der Antike*, pp. 76–80 and T. L. Heath, *History of Greek Mathematics*, I, pp. 246–249.

⁶ Becker (*op. cit.*, p. 79) presents this curve via a somewhat modified construction and cites its use by Villapando, as reported by Viviani (1647); see also G. Loria, *Ebene Kurven*, 1902, p. 317. Kepler's use of the same curve is noted by W. Breidenbach, *Das delische Problem*, Stuttgart, 1953, pp. 31 f.

⁷ For this part of the analysis, one may compare the account by van der Waerden. The principal difference lies in that I seek a determination of the locus of point I as a result of the analysis, while van der Waerden introduces it as an unexplained intuitive insight at the start.

⁸ Eutocius, *op. cit.*, p. 56.

⁹ See, for instance, Heath, *History*, I, p. 249; and I. Thomas, *Greek Mathematical Works*, I, p. 260n.

¹⁰ P. Tannery, "Lignes courbes," *Bulletin des sciences mathématiques*, 1884, 19, p. 101; cf. also *Mémoires de la Société des sciences physiques et naturelles de Bordeaux*, 2, pp. 277–283. Accounts appear in Heath, *History*, I, pp. 249–251; and Becker, *op. cit.*, pp. 78–80. Breidenbach also notes Descartes' use of this curve (*op. cit.*, pp. 29 f); cf. the *Géométrie* (ed. Smith and Latham), pp. 154–157.

¹¹ One may note that the curve generated via the condition $AR = AS$ [Fig. 6(a)] is extendable beyond the range of the intersection of the torus and the cone (i.e., up to the point where $AR = AB$). Setting $PS = y$, $AS = x$, and $AN = a$, one obtains $y : x = \sqrt{x^2 - a^2} : a$, or $a^2 y^2 = x^2(x^2 - a^2)$. The curve extends to infinity with increasing slope as x increases, becoming asymptotic to the parabola $ay = x^2$. Its inflection point V lies at the intersection of the line $x = Aa = \sqrt{2}a$ and the circle of diameter $Ac = \frac{1}{2}\sqrt{2}a$, so that $AV = \frac{1}{2}a$. (Fig. 6(b)) V here corresponds to I in the construction of the two mean proportionals AK, AI between AD, AB , for $AD : AB = \sqrt{3}\sqrt{3} : \sqrt{2}\sqrt{2}$. For $Ab : AN = 3 : 2$, V will correspond to T, the terminus of intersection of cone and torus. The problem of doubling the cube corresponds to $Ae : AN = 4 : 1$.

¹² R.C. Riddell, "Eudoxan Mathematics and the Eudoxan Spheres," *Archive for History of Exact Sciences*, 1979, 20, pp. 1–19.

¹³ In the modern theory of curves, this is termed the "spherical lemniscate"; projected onto the plane tangent to the cylinder at the double point, it becomes the Bernoullian lemniscate; cf. C. Zwikker, *Advanced Plane Geometry*, Ch. XVII. For its relation to the sections of the torus, see Chapter 6.

¹⁴ This sketch is not intended to be self-sufficient. For details, consult Riddell. Other treatments are given by O. Neugebauer, *Exact Sciences in Antiquity*, pp. 153 f, 182 f; van der Waerden, *op. cit.*, pp. 180–182; and Heath, *History*, I, pp. 329–335.

¹⁵ Detailed critiques of Eudoxus' astronomical theory are given by Neugebauer, *History of Ancient Mathematical Astronomy*, pp. 677–685; Heath, *Aristarchus*, Ch. 16; and L. E. J. Dreyer, *History of Astronomy*, . . . , Ch. IV.

¹⁶ Cf. Proclus, *In Euclidem*, pp. 112, 127 f.

¹⁷ Drawings of this curve are provided by Riddell, *op. cit.*, Figs. 12, 14.

¹⁸ Under this condition, the projection curve is a two-cusped epicycloid, as shown in Fig. 8(b), and is related to the epicyclic curves discussed in Chapter 7. When OG is less than GQ , the curve will have two loops in place of the cusps, and when OG is greater than GQ , the curve will arch smoothly inward at these positions. Riddell's Fig. 11 illustrates a curve of the latter form.

¹⁹ This is Riddell's solution method. But it leads to an alternative which dispenses with reference to the rotating spheres. If we conceive of the plane curve as generated by the epicyclic motion of circle JRH , then the perpendicular from T will meet it in R, while the generating circle will have assumed the position JRH ; thus, OJ is determined, from

which the construction of OS, OV follows. If we admit the epicyclic curve alone, without reference to the continually changing positions of circle JRH, nevertheless, the specific position of that circle corresponding to the point of intersection R can be produced by a planar construction; for the circle will pass through a given point and be tangent to two given circles. The latter problem falls under one of the classes of planar constructions treated by Apollonius in his *Tangencies*; cf. Chapter 7.

²⁰ But one may note that the modified Eudoxean scheme which Schiaparelli would assign to Callippus introduces the sphere of double speed, as in Riddell's construction. The three-sphere arrangement, however, produces a more complicated curve in which the figure-8 obtains additional double loops at each extremity. For a discussion, see Heath, *Aristarchus*, pp. 212–216.

²¹ Eutocius, *op. cit.*, pp. 56–58.

²² See the discussion in Chapter 2. According to the lexicon of Suidas, Eratosthenes was known as "a second (*deuteros*) or new (*neos*) Plato"; cf. I. Thomas, *op. cit.*, II, p. 260.

²³ See, in particular, *Republic* 510 c–e, 525 ff.

²⁴ This is comparable to the view proposed by van der Waerden, *op. cit.*, pp. 159–165. He does not actually attribute the pseudo-Platonic construction to Eratosthenes; but that appears to be implied in his account.

²⁵ Cf. Loria, *Ebene Kurven*, I, p. 50. A more detailed discussion appears in Chapter 6 (note 107).

²⁶ Banū Mūsā, *Verba filiorum*, Prop. 17; cf. M. Clagett, *Archimedes in the Middle Ages*, I, pp. 340–344. Further discussion will appear in the sequel. Only the geometric form of the construction is given in the Latin version of this work (made by Gerard of Cremona in the 12th century). But the Arabic version, extant in the recension by al-Tūsi (13th century), adds to this the details of the physical mechanism by which the construction can be produced.

²⁷ *Verba filiorum*, Prop. 16; cf. Clagett, *op. cit.*, pp. 334–340.

²⁸ Note, for instance, the expressions "to the extent that . . ." (*epi toouton mechris an*) and "the beam has the position such as has . . ." (*ton kanona thesin echein hoian echei . . .*) used here by Eutocius (*op. cit.*, p. 58) and their reappearance in his versions of the methods of Philo and Apollonius (*ibid.*, pp. 62, 64, 66). His mechanical terminology parallels that in his texts of Nicomedes (*ibid.*, pp. 98–100) and of Pappus (*ibid.*, p. 70; cf. *Collection* VIII, p. 1070), less that in his text of Eratosthenes (*ibid.*, pp. 92–94).

²⁹ Eutocius, *op. cit.*, pp. 78–84.

³⁰ See, for instance, H. G. Zeuthen, *Lehre von den Kegelschnitten*, Ch. 21, especially pp. 457–466 and the synopsis of "Menaechmus' probable procedure" by Heath, *History*, I, pp. 251–255 and II, pp. 110–116 and his *Apollonius*, Ch. 1.

³¹ After sketching Zeuthen's reconstruction of a Menaechmean derivation of the asymptote property of the hyperbola, Coolidge observes: "I confess to finding it a bit difficult to believe that Menaechmus could work out all this. But we are forced to accept something of the sort if we are to credit him with the discovery of the conic sections in connexion with the problem of duplicating the cube" (*History of the Conic Sections*, p. 5). Such rare, but well-founded misgivings are set at rest in the view proposed below.

³² Both relate to the elliptical appearance of circles viewed obliquely. For discussion, see Chapter 4.

³³ For accounts of the early theory, see Zeuthen, *op. cit.*, Ch. 2, 19, 20, and Heath, *Apollonius*, Ch. 3. A compilation of Archimedean theorems on conics is given by Heiberg in "Die Kenntnisse des Archimedes über die Kegelschnitte," *Zeitschrift für Mathematik und Physik*, hist.-litt. Abth., 1880, pp. 41–67.

³⁴ Eutocius, *op. cit.*, p. 96.

³⁵ Note that the curves discussed above as reconstructions proposed for Eudoxus' method rely on pointwise procedures, as do several of the accounts of other curves, like the quadratrix; cf. van der Waerden, *op. cit.*, p. 192.

³⁶ See *Diocles*, ed. Toomer, Props. 12 and 4, respectively. Eutocius (*op. cit.*, pp. 66–70) presents a version of the former in which the points are joined by rectilinear elements, rather than by means of a flexible ruler. But the flexible ruler is clearly indicated in Diocles' own account; cf. Toomer, pp. 159 f.

³⁷ Eutocius in Apollonius, *Opera*, ed. Heiberg, II, pp. 230–234.

³⁸ See Heiberg, *Mathematici Graeci Minores*, 1927, pp. 78 ff. 85 ff. These constructions are discussed in Chapter 6.

³⁹ Diocles, *op. cit.*, Prop. 10.

⁴⁰ Proclus, *In Euclidem*, p. 419. Plutarch, *Moralia*, 720 a, 1094 b; see Heath, *History*, I, pp. 144 f. For accounts of this technique, see Heath, *ibid.*, pp. 150–154; van der Waerden, *op. cit.*, pp. 118–126; and Becker, *op. cit.*, pp. 60–64.

⁴¹ In Fig. 13(a), if $CD = X$, $DF = A$, E bisects DF , and $DH = B$, then the "hyperbolic" case makes DH the mean proportional between CD and $DG (= A + X)$. This gives rise to the right triangle EDH having two given sides $ED (= \frac{1}{2} A)$ and DH ; thus, $EH (= \frac{1}{2} A + X)$ is also given, from which CD can be solved. Similarly, in Fig 13(b) the "elliptic" case makes DH the mean proportional between CD and DF , whence $ED (= \frac{1}{2} A - X)$ is given from the triangle EDH . For the relation between these analyses and the syntheses given by Euclid, see Heath, *Euclid's Elements*, I, pp. 382–388 and II, pp. 260–267; and Becker, *op. cit.*, pp. 62 f.

⁴² See Neugebauer, *Mathematical Cuneiform Texts*, 1945, and his *Exact Sciences in Antiquity*, pp. 40–42; examples are given by van der Waerden, *op. cit.*, pp. 63–71, and their relation to the Greek technique is elaborated, *ibid.*, pp. 122–124.

⁴³ On the manner of this transmission, see my "Techniques of Fractions." Neugebauer remains vague (*Exact Sciences*, pp. 149–151, 181 f.), although he insists on a real link between the two traditions.

⁴⁴ *Collection* (III), I, pp. 132–162. This is not to suggest that Pappus' treatment has a pre-Euclidean provenance. These constructions may well have their origin with Aristaeus in the time of Apollonius; cf. Chapter 7.

⁴⁵ *In Euclidem*, p. 67. For an interpretation of this reference to Eudoxus, see my *Evolution*, Ch. VIII.

⁴⁶ See Chapter 2, especially note 88.

⁴⁷ For further discussion, see my *Evolution*, Ch. II/II, VI/IV.

⁴⁸ This results easily from an analysis based on "completion of the square." Assuming $BF^2 = FE \cdot BE$, one has $BF^2 + BF \cdot BE = BE^2$, or $(BF + \frac{1}{2} BE)^2 = BE^2 + (\frac{1}{2} BE)^2$, so that $BF + \frac{1}{2} BE$ is the hypotenuse of a right triangle of sides BE , $\frac{1}{2} BE$. Setting $BG = \frac{1}{2} BE$ and $HG = EG$, it follows that $BF = HB$ is the required length. The corresponding synthesis is given in Euclid's II, 11. This form of analysis has been

proposed as underlying some of the computations preserved in Mesopotamian tablets from the mid-2nd millennium B.C.; cf. van der Waerden, *Science Awakening*, pp. 69 f. It thus contributes to the view of the transmission of the older technique to the Greeks during the pre-Euclidean period cited in note 43 above.

³⁰ R. Fischler proposes some further views on these early studies in "A Remark on Euclid II, 11," *Historia Mathematica*, 1979, 6, pp. 418–422.

³¹ One must avoid assigning too much weight to this form of negative testimony, however. Euclid's work is not an exhaustive repository of the geometric knowledge of his time, and one must allow for the prior availability of alternative methods from among which Euclid could choose for the proofs in the *Elements* (cf. my *Evolution*, Ch. I/II). Nevertheless, it is always preferable, when engaging in reconstruction, to have some positive evidence from the surviving documents.

³² See the discussion of the Aristotelian and Apollonian locus problems in Chapter 4; the related problem preserved by Pappus (Chapter 4, note 16); and the *neuses* from Heraclitus and Apollonius (Chapter 7).

³³ Euclid adopts an oddly roundabout method here, taking angle EFA as an exterior angle to triangle CFA. One could proceed more directly by considering the triangles AFE, CAE which have equal angles at C and A and share the angle at E; thus, the remaining angles EFA, EAC are equal.

³⁴ Heath reports just such a reconstruction, taken from Todhunter; cf. *Euclid's Elements*, II, pp. 99 f.

³⁵ See also the translations by Heath, *History*, I, p. 299; and Thomas, *op. cit.*, I, pp. 394–397.

³⁶ This was proposed by August (1829) and Butcher (1888) and adopted by Heath, *History*, I, pp. 299–303. For a detailed survey of these and many other proposals, see R. S. Bluck, *Plato's Meno*, 1961, pp. 441–461. Bluck discusses the general method in this passage in relation to geometrical analysis, *ibid.*, pp. 76–85.

³⁷ So Heijboer, cited by Bluck, *op. cit.*, p. 448. Cf. Heath's response to the objections by Benecke, *History*, I, p. 302.

³⁸ Bluck, *op. cit.*, Introduction, Ch. E, esp. pp. 118–120 reviews the arguments for the dating of the *Meno*.

³⁹ We provide an elementary proof that the equilateral triangle is the greatest of all triangles inscribed in the same circle. Since of all triangles on a given chord AB [Fig. 19(a)] the isosceles triangle CAB is the greatest, we have only to compare the equilateral with the isosceles cases. First consider the isosceles triangle ADE [Fig. 19(b)] whose base DE is parallel to and shorter than a side BC of the inscribed equilateral triangle ABC. Since the bisector H of arc ADE lies on arc AB, triangle ABE will be greater than ADE; for the altitude BF of the former will be greater than the altitude DG of the latter. From the previous case, the equilateral triangle ABC will be greater than ABE, since the former is isosceles with respect to the common base AB. Hence, *a fortiori*, ABC is greater than ADE. Similarly, if side DE is greater than BC [Fig. 19(c)], ABE will again be greater than ADE, for here the bisector of arc ADE will lie on arc BC. It follows as before that the equilateral triangle ABC is greater than ADE. We have thus established that the equilateral triangle is greater than any other triangle inscribed in the same circle. It seems to me that no feature of this proof of the diorism would have posed any difficulty for a Greek geometer. Thus, the technical effort from which Plato draws here must surely

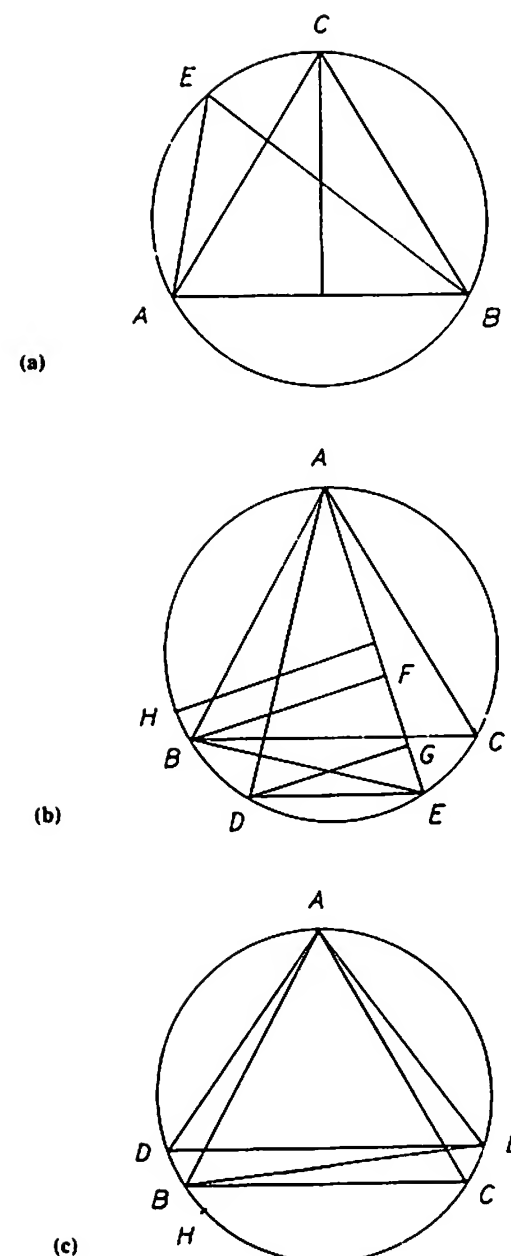


Figure 19

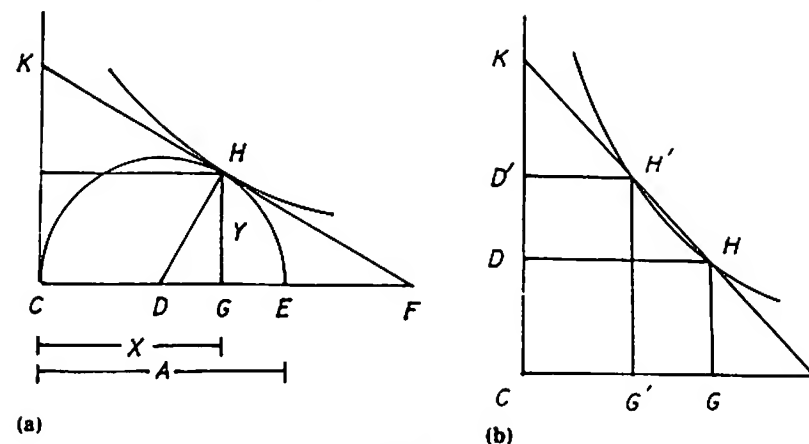


Figure 20

have been directed toward the explicit construction of the inscribed triangle of given area. Our good fortune in learning of this effort through this passage thus owes to Plato's slip in phrasing the diorism in accordance with the terms of the construction problem itself, thus losing sight of its independent treatment via elementary methods.

⁷⁶ In Fig. 20(a), setting $HG = Y$, $CE = A$, and $CG = X$, we have, from the property of the tangent to the hyperbola, $FH = HK$, whence $FG = GC$ (cf. Chapters 5, 6, 7). Since HG is the mean proportional of FG , GD and also of EG , $GC (= FG)$, it follows that $DG = GE$, so that $X = \frac{1}{2}A$. The property of the tangent can readily be deduced from the constant-product property of the curve. Let the curve be figured with respect to the lines FC , CK such that for any points H , H' on it, $GH \cdot HD = G'H' \cdot H'D'$ [Fig. 20(b)], and let the chord HH' meet CF , CK at F , K , respectively; it is claimed that $FG = D'H'$. For $HD : H'D' = G'H' : GH$; by similar triangles, $G'H' : GH = FG' : FG$. Thus, $HD - H'D' : H'D' = FG' - FG : FG$. Since $HD - H'D' = FG' - FG (= GG')$, $H'D' = FG$, as claimed. If the secant is now adjusted so that H' approaches H , in the limiting position of the tangent one has $FG = GC$, as required above. One may note that this derivation effectively reverses the order followed in Apollonius' *Conics*, for there the constant-product property of the hyperbola (II, 12) follows after the equality of the segments of the secant, $FH = H'K$ (II, 8). As it happens, Apollonius does not use II, 8 for the proof of the lemma II, 10 (that $FH \cdot HK = FH' \cdot H'K$) on which II, 12 depends. This odd isolation of II, 8 (and its corollary tangent theorem, II, 9) doubtless signifies stages in the composition and transmission of the *Conics*; see Chapter 7.

⁷⁷ In *Euclid*, pp. 220 f. This passage is discussed by Heath, *Euclid*, I, p. 128.

⁷⁸ A similar instance is cited by Pappus as an example of an "indeterminate" (*adioriston*) problem: the application with square excess over a given line segment yields widths arbitrarily large or small, depending on the magnitude of the area applied (*Collection* (VI), II, p. 542). Thus, the "unordered" problems of Amphinomus are parallel to the "indeterminate" of Pappus, in that each refers to problems having separate solutions for each choice of a parameter from a specified range. Menaechmus' curves for the cube duplication correspond precisely to this class of problems.

⁷⁹ In *Euclid*, p. 202. A puzzle here is that Aristotle *does* maintain that demon-

strations explain the causes of propositions; cf. *Posterior Analytics* I, 2, 71 b 8 - 32; and 24, 85 b 23 - 28. The difficulty is noted by G. Morrow, *Proclus: A Commentary* . . . , 1970, p. 158 n.

⁸⁰ In *Euclid*, pp. 77 f. There would seem to be a discrepancy with Proclus' ascription to Amphinomus of a classification of problems (see note 61 above). But perhaps a commitment to the priority of theorems over problems need not entail the view that the format of problems should be eliminated altogether.

⁸¹ 78 a 7 - 14; cf. also *Prior Analytics* II, 2-4. Other discussions of analytic method in the more general dialectical context appear in *Nic. Eth.* 1112 b 11 - 24 and *Post. Ana.* I, 22, 84 a 6 ff. Such passages are examined by J. Hintikka and U. Remes, *The Method of Analysis*, Ch. 2, in the context of Pappus' account of this method (*Collection*, VII, Preface).

⁸² This passage correctly understands analysis as the examination of the deductive consequences of the hypothesis; other passages in Aristotle and later writers (see the references in note 64) attempt to construe it alternatively as a search for appropriate antecedents, but this confusing feature conflicts with the unanimous testimony of the mathematical evidence. It would also render superfluous any concern over convertibility, since the analysis (of antecedents) would of itself have produced the deductive sequence of the synthesis. I shall not attempt to press further these early passages on the nature of analysis, but the reader who wishes to do so may consult J. Barnes, "Aristotle, Menaechmus, and Circular Proof," *Classical Quarterly*, 1976, 26, pp. 278-292 for a more ambitious interpretation.

⁸³ *Soph. Ref.* xi and *Prior Ana.* I, 9. For discussion see Heath, *Mathematics in Aristotle*, pp. 47-50.

⁸⁴ Philoponus, *In Ana. Post.*, ed. Wallies, pp. 111-114 cites views of Alexander, Ammonius, and Proclus concerning Bryson's argument. Simplicius' long passage on the circle quadrature does not consider Bryson (*In Physica*, ed. Diels, I, pp. 53-69).

⁸⁵ The specification of *squares* appears only in the comment by pseudo-Alexander (*In Soph. El.*, ed. Wallies, p. 90); on this basis one has been led to speculate whether Bryson intended that the square formed as the arithmetic or the geometric mean of the constructed squares ought to equal the circle. But in the paraphrase by Philoponus, Alexander considers an unrestricted "rectilinear figure," thus indicating a more sophisticated view.

⁸⁶ So Proclus, as cited by Philoponus, *op. cit.*, pp. 111 f. Proclus objects to Alexander's interpretation on the grounds that it would make the fallacy of Bryson's argument the same as that of Antiphon's: that the circle will coincide with the straight line. Proclus thus seems to have in mind the case where the inscribed and circumscribed figures are so close to each other that the intermediate polygon becomes indistinguishable from either and hence coincides with the circle. This need not be rejected as a possible view of Bryson's argument, even if Proclus himself goes on to propose an alternative.

⁸⁷ *Op. cit.*, pp. 112 f. The principle already appears in the Aristotelian *Mechanics* (847 b 25): that whatever changes from one extreme to the other (e.g., from convex to concave, or from great to small) must sometime be in the mean state (e.g., straight or equal). It may well be rooted in Academic discussions about the greater and lesser, the limit and the unlimited, such as in Plato's *Philebus* (23 c - 27 c).

⁸⁸ *Ibid.*, p. 112. Cf. Proclus, *op. cit.*, p. 234, who observes that in the case of the

horned angle, one does not pass from the lesser to the greater through the equal; that is, no rectilinear angle equals the curvilinear.

⁷² Mixed angles are introduced in the pseudo-Euclidean *Catoptrics*, Hero's *Catoptrics*, Euclid's *Optics* in the recension by Theon, and the Bobbio mathematical fragment (discussed in Chapter 6, note 99). For references, see Toomer, *Diocles*, pp. 156 f and my "Geometry of Burning-Mirrors in Antiquity."

⁷³ *Op. cit.*, p. 112.

⁷⁴ Archimedes proves this in a more sophisticated manner in *Sphere and Cylinder* I, 5, 6.

⁷⁵ So Heath, *Mathematics in Aristotle*, pp. 48 f. See also Becker, "Eudoxos Studien II" and "III" (*Quellen und Studien*, 1933 and 1936, respectively) and Ian Mueller, "Aristotle and the Quadrature of the Circle" in *Infinity and Continuity in Ancient and Medieval Thought*, ed. N. Kretzmann. Mueller quite plausibly injects a kinematic element into the conception of the sequence of squares.

⁷⁶ For instance, the existence of the fourth proportional is regularly assumed in the limiting theorems of Euclid and Archimedes, doubtless in agreement with Eudoxus' procedure (see discussion below). See also Becker, "Eudoxos-Studien II" and my "Archimedes and the Pre-Euclidean Proportion Theory." Mueller points to this as a characteristic nonconstructive feature in Euclid's geometric methods; see his *Philosophy of Mathematics . . . in Euclid's Elements*, 1981, Ch. 3.2, 6.3.

⁷⁷ On Archimedes' use of Eudoxean techniques, see my "Archimedes and the *Elements*" and "Archimedes and the Pre-Euclidean Proportion Theory." For accounts of Eudoxus' method of limits, see van der Waerden, *Science Awakening*, pp. 184–187 and Mueller, *Philosophy of Mathematics*, Ch. 6.3.

⁷⁸ Euclid proves this by considering the rectangle enclosing each segment on the sides of the inscribed polygon. Since the inscribed triangle equals exactly one-half the rectangle, while it, in its turn, is greater than the segment, the triangle is greater than one-half the segment. Hence, the passage from one polygon to the next will remove more than one-half the difference in area between the circle and the initial polygon.

⁷⁹ This follows from the lemma (X, 1) on the indefinite diminution of a finite magnitude through successive bisection.

⁸⁰ Via this ploy, Euclid eliminates the need for circumscribed figures in all the limiting theorems in Book XII (i.e., Props. 2, 5, 10, 11, 12, 18). This feature may originate with Eudoxus' treatment of these theorems. But it is just the sort of refinement which one would expect from a subsequent rigorizing editor, for instance, a follower of Eudoxus or Euclid himself. One need not deny then that Eudoxus might have employed circumscribed figures in his treatment. In this case, Archimedes could have seen in Eudoxus' work a precedent for his own use of such figures, as in the *Dimension of the Circle*; see Chapter 5.

⁸¹ See Chapter 2.

⁸² This was noted by Becker in his "Eudoxos-Studien IV" (*Quellen und Studien*, 1936). For a discussion of its implications for understanding Eudoxus' form of proportion theory, see my "Archimedes and the Pre-Euclidean Proportion Theory."

⁸³ See my "Archimedes and the *Elements*" and Chapter 5.

⁸⁴ A more elaborate reconstruction is proposed by Becker in his "Eudoxos-Studien

IV." For accounts of the Euclidean technique of proportions, see Heath, *Euclid*, II, pp. 116–131; van der Waerden, *op. cit.*, pp. 187–189; Mueller, *op. cit.*, Ch. 3; and my *Evolution of the Euclidean Elements*, Ch. 8 and Appendix B.

⁸⁵ One can readily provide a constructive test for the inequality $A : B > n : m$. If a sequence of regular polygons P is inscribed in circle A , while a sequence of similar regular polygons Q is circumscribed about B , then one will eventually have $P : Q > n : m$, whence *a fortiori* $A : B > n : m$. A similar test applies in the case of the opposite inequality.

⁸⁶ This may be done by the bisection method, as in XII, 2, or by the alternative method of concentric circles used by Euclid for the volume of the sphere; cf. XII, 16.

⁸⁷ This view is elaborated in my "Archimedes and the Pre-Euclidean Proportion Theory."

⁸⁸ See Becker, "Eudoxos-Studien II." For other instances one may note Archimedes' *neuses* in *Spiral Lines*, and indeed *all* applications of *neuses*; also, the assumption of points of intersection of given lines and curves (cf. Heath, *Euclid*, I, pp. 234–240).

⁸⁹ "Die geometrische Construction als 'Existenzbeweis' in der antiken Geometrie," *Mathematische Annalen*, 1896, 47, pp. 222–228. For a criticism of this view in relation to the Euclidean geometry, see Mueller, *op. cit.*, pp. 15, 28.

⁹⁰ *Collection* (IV), I, pp. 250–252; Iamblichus is cited by Simplicius, *In Physica*, I, p. 60.

⁹¹ *In Euclidem*, pp. 272, 356.

⁹² This is a view actually maintained by some; cf. K. Freeman, *Pre-Socratic Philosophy*, Oxford, 1946, p. 355; and W. D. Ross, "Hippias of Elis" in the *Oxford Classical Dictionary*, 2nd ed., p. 517.

⁹³ Tannery, "Lignes courbes," 1883, pp. 278–281; Björmbom, "Hippias," in *Paulys Wissowa*, 8, 1913, col. 1708 f; Heiberg, *Geschichte der Mathematik . . . im Altertum*, p. 5; Heath, *History*, I, pp. 23, 225 f. Apparently, these writers maintained this as the "traditional" view, and so felt little need actually to argue a textual case for it. Cantor's line is typical: this Hippias must be Hippias of Elis, for the ancient tradition preserves no mention of any other geometer of this name. But, of course, the tradition preserves merely single references to many geometers (as a glance at the name-indices to Pappus, Proclus, and Eutocius reveals); while we may easily estimate that dozens have been forgotten altogether. Cantor also begs the extremely important question as to whether the Sophist was a geometer, and so must invent reasons for his being omitted from Proclus' list of Euclid's precursors (*In Euclidem*, pp. 65–67).

⁹⁴ Hankel, *Geschichte der Mathematik*, 1874, p. 151n. Allman initially accepts Hankel's view (*Greek Geometry*, 1889, pp. 92–94), but is later persuaded by Cantor's line and so adopts a position midway between Hankel and Tannery: that Hippias did indeed introduce the curve, but for angle trisection rather than circle quadrature (*ibid.*, pp. 93n, 189–193). This is the view held by van der Waerden (*op. cit.*, pp. 146, 191) and by Becker (*op. cit.*, pp. 95 ff); but Guthrie seems unconvinced (*History of Greek Philosophy*, III, pp. 283 f).

⁹⁵ *In Euclidem*, p. 65.

⁹⁶ For a survey, see Guthrie, *op. cit.*, pp. 280–285.

⁹⁷ *Hippias Minor* 367 c–368 e. In the *Protagoras* (e.g., 315 c, 318 e) Hippias appears

as a teacher of the mathematical sciences and natural philosophy, but no specific indication is given of his geometric expertise.

⁹⁸ In *Euclidem*, p. 356.

⁹⁹ Eighteen figures bearing this name are noted in *Pauly Wissowa*, Vol. 8.

¹⁰⁰ A more detailed discussion is given in Chapter 6. For other technical accounts, see Heath, *History*, I, pp. 226–230; van der Waerden (on Dinostratus), *op. cit.*, pp. 191–193; and Becker, *Denken*, pp. 95–98.

¹⁰¹ Cf. *Elements* VI, 11.

¹⁰² Cf. Archimedes, *Dimension of the Circle*, Prop. 1.

¹⁰³ *Collection* (IV), I, pp. 250–262; the proof is given in pp. 256–258.

¹⁰⁴ E.g., that similar arcs have the ratio of the diameters of their respective circles. For a discussion of this theorem and its relation to the extant *Dimension of the Circle*, see my “Archimedes and the *Elements*” and “Archimedes and the Pre-Euclidean Proportion Theory.” Note that this use of the quadratrix also requires the equivalent of the inequality $\sin x < x < \tan x$; the first known instance of this is in Archimedes’ *Dimension of the Circle*, Prop. 1.

¹⁰⁵ Cited by Pappus, *Collection* (IV), I, pp. 252–256. The difficulty is examined in the accounts by Heath and van der Waerden (see note 100 above).

¹⁰⁶ From *symbolleîn*, *asymblêta* is often rendered as “incommensurable” (for which, however, the usual term is *asymmetros*) or as “incomparable,” that is, literally, “unable to meet.” Aristotle doubtless has in mind the fact that circles and lines cannot be made to coincide with each other. This would bring this passage into the context of Antiphon’s and Bryson’s circle quadratures; cf. note 69 above and Chapter 2.

¹⁰⁷ *Physics* 248 a 19 – b 7; cf. Heath, *Mathematics in Aristotle*, pp. 140–142. The elliptical phrasing is quite typical of Aristotelian writing, but can be supplied with little fear of ambiguity.

¹⁰⁸ See, for instance, Aristotle’s discussions of the Eudoxean and Callippean planetary systems in *Metaphysics* XII, 8.

¹⁰⁹ See, for instance, Pappus, *Collection* (IV), I, p. 258, (V) pp. 312 ff, and (VIII), p. 1106; also In *Ptolemaeum*, ed. Rome, 1931, pp. 254 ff. Hero, *Metrica* I, 26 (cf. I, 37). Theon, In *Ptolemaeum*, ed. Rome, 1936, pp. 359 ff (here following Zenodorus; see Chapter 6). Proclus, In *Euclidem*, p. 423. One may of course add Eutocius’ commentary on *Dimension of the Circle*, *op. cit.*, pp. 228 ff. The writers and scholiasts in the metrical tradition growing out of Hero’s work concur in the Archimedean attribution; cf. Hero, *Opera*, ed. Heiberg, IV–V.

¹¹⁰ The elementary character of Archimedes’ method here is one of the indications I employ for arguing an early dating for this work; see my “Archimedes and the *Elements*.”

¹¹¹ *Collection* (IV), I, pp. 250–252. Note that the word for “received” (from *paralambaneîn*) would regularly mean “received (from)”, as in the manner of inheritance. I suggest reading “apo” (“from”) for the manuscript’s “hypo” (“by”), so to obtain “a curve received from Dinostratus *et al.*” This would have the effect of removing the implication made in the passage as it now stands, that Dinostratus introduced the curve in the context of circle quadrature.

¹¹² *Collection* (IV), I, pp. 284–286; the spiral is used for the same purpose, *ibid.*, pp. 286–288.

¹¹³ See Archimedes’ *Spiral Lines* and Pappus, *Collection* (IV), I, pp. 234 ff. This is discussed in Chapter 5.

¹¹⁴ *Collection* (IV), I, pp. 270–272.

¹¹⁵ That is, by means of the “Euclidean” constructions using compass and straightedge alone.

¹¹⁶ In *Euclidem*, pp. 66 f.

¹¹⁷ For a discussion of these studies, see my *Evolution*, Ch. 8.

¹¹⁸ I examine several views of this sort in my “On the Early History of Axiomatics” and “Infinity and Continuity.”

¹¹⁹ One perceives such a tendency in Heath’s accounts and those he cites; see his *History*, I, pp. 222, 224, 196n.

¹²⁰ The chief proponent of this view is S. Luria, “Die Infinitesimaltheorie der antiken Atomisten,” *Quellen und Studien*, 1933, B : 2, pp. 106–185; it is elaborated by J. Mau, *Zum Problem des Infinitesimalen bei den antiken Atomisten*, Berlin, 1954. Brief accounts appear in van der Waerden, *op. cit.*, pp. 137 f; and Guthrie, *op. cit.*, II, pp. 487 f. See also Heath, *Euclid’s Elements*, III, pp. 366–368.

¹²¹ The thorough study by H. J. Waschkiess of the roots of Aristotle’s continuum theory (*Von Eudoxos zu Aristoteles*), I believe, pays insufficient attention to the technical sources of Eudoxus’ geometric methods. By contrast, Mueller maintains that no specific philosophic motive need be invoked for understanding the methods of Eudoxus and Euclid; cf. *op. cit.*, pp. 10, 234.

¹²² For surveys of the background of the modern studies of foundations, one may consult N. Bourbaki, *Éléments d’histoire des mathématiques*, “Fondements des mathématiques,” pp. 9–63; and M. Kline, *Mathematical Thought from Ancient to Modern Times*, 1972, Ch. 40–43, 51.

¹²³ For references, see my *Evolution*, Ch. 9 and the papers cited in note 118 above.

¹²⁴ See his “Eudoxos-Studien I–IV,” *Quellen und Studien*, 1933–36.

¹²⁵ See my discussion in *Evolution*, Ch. 2.

¹²⁶ See Chapter 2.

CHAPTER 4

The Generation of Euclid

With specific reference to the three "classical" problems our sources preserve no solving efforts from the half-century or so separating the successors of Eudoxus from Archimedes. Nevertheless, this intermediate period, the generation of Euclid in Alexandria,¹ witnessed important advances in the field of geometric problem solving. At least five works by Euclid alone bear directly on the implementation of analytic efforts, while many of the more advanced problems which would concern geometers later in the 3rd century were defined and their analysis initiated at this time.

The *Elements* of Euclid are justly described as the most widely known and used technical textbook ever written.² In this work Euclid gathered the separate fields of elementary plane and solid geometry and number theory and edited prior treatments to produce a comprehensive and generally uniform compilation. Here the efforts of Eudoxus and his disciples in the field of "exhaustion" measures of curvilinear figures are gathered, as well as their development of the theory of proportions of magnitudes; the researches of Theaetetus and his successors in the fields of irrational lines and the construction of the regular solids are presented, together with the treatment of number theory on which they depend. These materials, which occupy somewhat more than half of the *Elements*, constitute fairly well circumscribed subjects whose study is of interest in its own right. But the remainder of the work is more elementary in character, intended to provide the basis of theorems necessary for the further study of geometry. Throughout, the treatment is synthetic, so that it sometimes is not immediately evident what the significance of a particular theorem might be. This

is true, for instance, of much of Book II, where one meets theorems having the appearance of auxiliary lemmas relating to a field often called "geometric algebra." These contain the proofs of theorems related to the techniques of the geometric application of areas which we discussed in connection with the cube duplication by Menaechmus.

Yet for all the sophistication of its logical structure and the intricacy of some of its constructions, especially with respect to the irrationals and the regular solids, the *Elements* is predominantly a treatise of an introductory sort, as its title implies. The researches assembled in it were initiated decades before Euclid by Hippocrates, Theaetetus, and Eudoxus and advanced by their successors. Even the organization into treatises comparable to the *Elements* had been pursued by several scholars in the middle and latter parts of the 4th century.⁴ Thus, not only is one hard put to identify any specific theorem in the *Elements* as the discovery of Euclid, but even his responsibility for the form of the proofs there presented and for the selection and ordering of the theorems in the separate books is quite in doubt. Clearly, the "Pythagorean theorem" was familiar a millennium before Euclid among the mathematicians in ancient Mesopotamia, for instance; but its proof in I, 47, which Proclus ascribes to Euclid himself, is patently a modification of that in VI, 31, and the result in this latter form can hardly be denied to Hippocrates, for it is fundamental for his quadrature of the lunules.⁵ Again, the Euclidean proportion theory in Book V without doubt incorporates modifications on the form of the theory initiated by Eudoxus; but several passages from Aristotle's discussions of the continuum make clear that such changes were already well under way three or four decades before Euclid.⁶ We may say the same for the theory of irrationals in Book X, or even that Euclidean principle *par excellence*, the postulate of parallels.⁷ The key conceptual insights as well as the major portion, if not the whole, of the work of securing them through formal proof had already been achieved by Euclid's predecessors.

Thus, the appropriate measure of the geometric researches conducted by Euclid and his contemporaries is to be sought not in the *Elements*, but in the *Data* and the lost *Porisms* and *Conics*, conceived for abetting the solution of geometric problems. But before describing these works, let us turn to the examination of a striking problem-solving effort representative of the new developments very near Euclid's time.

A LOCUS PROBLEM IN THE ARISTOTELIAN CORPUS

An odd stroke of good fortune has preserved for us a fine example of the synthesis of a locus problem from late in the 4th century: it is embedded as a lemma within the discussion of the shape of the rainbow in the Aristotelian *Meteorologica* (III, 5) and reads thus:

since the points K, H have been given, so also would the (line) KH be given, and also the (line) MH (is given), so that also the ratio of MH to MK. 'Thus M touches a given circumference.'

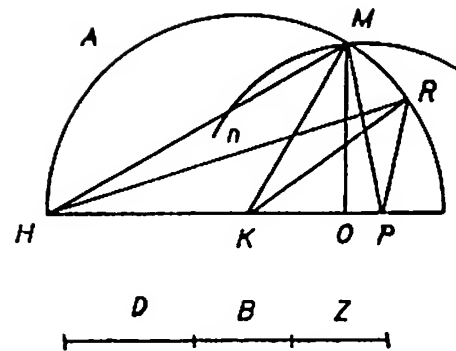


Figure 1

In the terminology of the "givens" (*data*) standard in the analytic literature, this expresses the result that the locus of points whose distances from two given points are in a given ratio (other than unity) is a given circle. This passage is unique in the Aristotelian corpus in that it provides a full geometric demonstration, and all the more remarkable in that the manner of proof is identical to that employed a century later by Apollonius in his second book *On Plane Loci*, by virtue of which the construction now goes under the name of the "Apollonian circles."⁹ Its proof should thus inform us of the technique adopted at an early phase of the problem-solving tradition.

The writer introduces a line segment and divides it into parts D and B such that $D : B = MH : MK$, the given ratio (see Fig. 1).¹⁰ He then requires that there be produced (*prospeporisthō*) an addition Z to B such that $D : B = B + Z : D$, and that accordingly, KH be extended to P such that $Z : KH = B : KP$. He joins MP and claims that "P will be the center (*polos*) of the circle toward which the lines from K fall," that is, the broken lines KMH satisfying the condition of the locus. Let the line PR be taken such that $Z : B : D = KH : KP : PR$, so that $PR : KP = PH : PR$. This means that the triangles PKR and PRH are similar, for they have the angle KPR in common. Thus, $PR : KP = HP : PR = D : B$, the given ratio. Hence, on the one hand, R is a point on the locus, while on the other, PR is the mean proportional of KP, PH whose magnitudes are given (that is, independent of the choice of R) so that PR is also given in magnitude. Hence, $PR = PM$ and the locus required is the circle of center P and radius PM.

This passage is studded with interpretive puzzles. First, there is the question of the relation between the Aristotelian and Apollonian versions of the proof. The constructions are evidently the same, but there are differences in the demonstrations. The Aristotelian form adopts an indirect proof, in which the auxiliary line PR is hypothesized to be greater or less than PM, and it is deduced that both $HR : RK$ and $HM : MK$ equal the given ratio $D : B$. But why this should be an apparent contradiction is not quite clear. Perhaps the writer has assumed the special context of his construction, where the trial point R is to be taken

along the circumference of the circle HAM; in this case it is indeed true that for PR less than PM, the ratio HR : RK must be greater than HM : MK, since HR will be greater than HM and RK will be less than MK. Otherwise, he seems to beg the question in supposing as obvious that if R does not lie on the arc Mn the ratio HR : RK must be different from the given ratio.¹¹ In the Apollonian version the argument is managed in direct fashion: the magnitude PM is not introduced, so that the deduction that points on the circle of radius PR satisfy the condition that HR : RK equals the given ratio D : B is precisely what has been sought, that this circle is the solving locus. In this way, the Apollonian version has considerably improved the logical structure of the proof. On the other hand, the Aristotelian version concludes at once that $PR : KP = HP : PR$ from the similarity of the triangles PKR, PRH. By contrast, the Apollonian version, despite awareness of the same pair of similar triangles, launches into an incredible argument of a dozen steps to deduce the same proportionality of lines. I find it hard to conceive how Apollonius could have missed the transparency of this step, and thus I would assign the longer detour to a later editor, possibly Eutocius, momentarily puzzled by a step assumed as obvious in Apollonius' proof.¹²

A second puzzle is that this construction seems to be more complicated than it need be. Simson has proposed a very neat alternative form which develops from the condition of the locus in a more natural manner.¹³ Consider what an analysis of the problem might be: given the points H, K, we choose any point M such that $HM : MK$ equals the given ratio. (See Fig. 2.) Divide HK at L so that the ratio of the segments HL : LK equals the same; then by *Elements* VI, 3, the line ML bisects the angle HMK. Now, extend HM beyond M to S and draw MT to bisect the angle SMK; one then has $TH : TK = HM : MK$.¹⁴ Thus, points L and T, as well as M, lie on the locus. Since ML and MT are the bisectors of supplementary angles, angle LMT is a right angle. We may thus draw the circle through L, M, T of diameter LT. As the points L, T are given via the given points and ratio which define the locus, and as L, T in their turn specify the circle of diameter LT, it follows that all points M of the locus lie on that circle. Thus, to construct the locus, one need merely determine the points L, T and draw the circle. This form of the solution is manifestly simpler than the

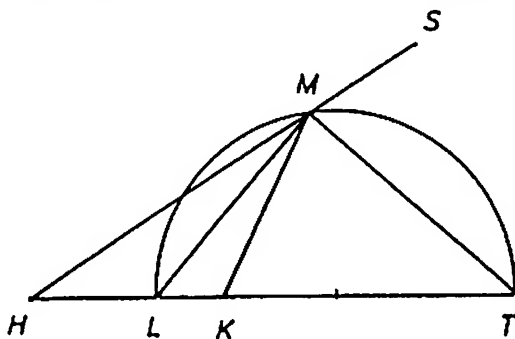


Figure 2

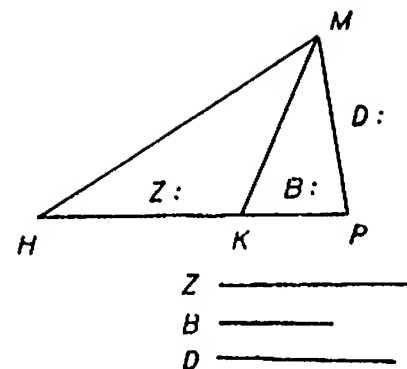


Figure 3

Aristotelian/Apollonian form. The contrast is sharpened when one observes that the points L, T do not enter into the latter form at all. Perhaps the originator of the ancient version did not have at hand the theorem on the bisector of the external angle, so critical for Simson's alternative. But one gains nothing in attempting to second-guess the ancient writer. What is striking is that Apollonius has sustained this relatively cumbersome approach to the problem. Not only is his dependence on a model essentially identical to the Aristotelian version thus affirmed, but also his treatment of earlier results is revealed. Apparently, his aim in the *Plane Loci* was not to redo the older solutions, but rather to collect these and use them as preliminary to the presentation of his own findings.

This also shows that the analysis implicit in the ancient construction of this locus could not have proceeded along the lines suggested by Simson.¹⁵ The implied analysis would appear to have taken the following form: given the points H, K, we assume the point M on the locus, so that $HM : MK$ equals the given ratio. [See Fig. 3] If we now extend HK and draw PM such that angle PMK equals angle KHM, then the triangles PMH, PKM will be similar, since they share the angle at P, while the angles at M and at H are equal. Thus, $HM : MK = PH : PM = PM : PK$, so that each equals the given ratio, and PM is the mean proportional of PH, PK. Consider lines Z, B, D such that $Z : B : D = HK : KP : PM$. Since $D : B$ is a given ratio, and D is the mean proportional between B and $B + Z$, it follows that Z is given from D and B. Since HK is also given, KP and PM are given in magnitude, for they are each in a given ratio to HK. Thus, the locus is seen to be the circle of center P and radius equal to PM. The construction and synthesis answering to this analysis are evidently those given in the Aristotelian/Apollonian solution; and in the presence of such an analysis, that solution loses all of the complexity which marks the synthesis when it stands by itself. Indeed, the construction is seen to hinge on a single insight, a rather clever one, of introducing the angle at M equal to that at H. In this way, Apollonius' decision to retain the older form, rather than search for a handier alternative, becomes easy to understand.¹⁶

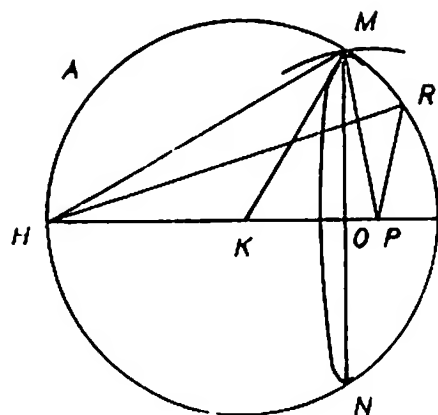


Figure 4

Our third puzzle is, why does this construction appear in the *Meteorologica* at all? For his explanation of the circular shape of the rainbow, the Aristotelian writer requires that the point M, marking the elevation of the bow above the horizon, be determinate. Upon rotation of the figure about the axis HKP, the point M will trace a circle, of which the half above the horizon defines the position of the whole bow (see Fig. 4).¹⁷ His stipulation for M is that it be the point on the given circle HAM for which the distances HM, MK are in a given ratio. This reasonably opaque assumption is not motivated through any physical considerations, such as the nature of the optical rays and their reflection; nor is any attempt made actually to specify the value of this ratio. Thus, we are hardly any better off than if he had simply stipulated that the angle PKM be a constant. Furthermore, he applies the condition in an unnecessarily complicated way. For, as we have seen, if M is taken on the circumference such that $HM : MK$ equals the given ratio, then for any point lower than M, the corresponding ratio will be greater than the given, while for any point higher than M, the ratio will be less. This is the view adopted by the later Aristotelian commentators, Alexander and Olympiodorus, who paraphrase in minute detail the entire construction and proof of the locus problem, although they have utterly failed to perceive its significance.¹⁸ These considerations make plain the derivative character of this context of the locus problem. The author of the *Meteorologica* has recognized that this construction offers one method for fixing the elevation of the rainbow. But the absence of the analysis leaves the construction without clear geometric motivation, while the stipulation of lines in given ratio stands without physical motivation. Moreover, the unnatural manner of deploying this condition within the special context of the passage—that is, presenting the general solution of the locus, rather than merely arguing the uniqueness of the point on the given circle satisfying the condition on ratios—seems to indicate that the writer was sufficiently impressed by this particular result to create a place for it within his

meteorological discussion. That is, it was taken to be a significant discovery, and that, we should suppose, was by reason of its being recent and representative of advanced study in geometry at the time.¹⁹

In speaking of the "Aristotelian writer," I betray a reluctance to take Aristotle as the author of this solution of the locus problem. It ranks among the finest samples of geometry we know of from the pre-Euclidean period, and nothing in the Aristotelian corpus prepares us for the notion that Aristotle himself was capable of such a technical feat.²⁰ The passage is singular on several accounts: only here and in a passage two chapters earlier in the same book of the *Meteorologica* does one come upon the presentation with full proof of a geometric proposition. The nearest parallels are a set of theorems on proportions involving infinite magnitudes given at several places in the *Physics* and *De Caelo* and effected with no remarkable skill or insight.²¹ Otherwise, Aristotle's mathematical passages either merely allude to aspects of proofs or take up the discussion of metamathematical issues. These are valuable as reflections of mathematical work of the period, but they provide no evidence of Aristotle's active participation in that work. Furthermore, one may well entertain suspicions about other features of the passage in the *Meteorologica*: the optical ray is conceived as emanating from the eye and outward to the object seen, here from K to H via reflection at M; this conforms with standard usage in ancient optical studies, but directly contradicts the theory of vision Aristotle favors against this view in *De Anima* (II, 7), in which seeing is an effect of the transparent medium acting on the eye, so that the account of vision harmonizes with the accounts of the other senses. Why should Aristotle not have adopted the same view in the *Meteorologica*? It is geometrically indifferent, and, indeed, optical writers sometimes do adopt it in a context, as here, where the object seen is a luminous body like the sun or a star.²² This discrepancy has long been noted by commentators, both ancient and modern; but it seems not to have provoked the uneasiness one should have expected.²³ Again, the analysis of the rainbow introduces the bizarre device of placing the sun (H) and the cloud producing the rainbow (M) at equal distances from the observer (K). This fiction, sometimes called the "meteorological hemisphere,"²⁴ is patently absurd from the cosmological standpoint, as Aristotle must have known, in view of his familiarity with the Eudoxean astronomy (in *Metaphysics* XII, 8) and his consistent distinction between the sublunary sphere of the elements and the higher domain of the heavenly bodies. Why should Aristotle here so casually evade the plain principles of cosmology? If one supposes that he merely adopts a standard meteorological convention, just as we saw before he followed the optical convention in the conception of the visual ray, then one must admit that Aristotle's thought has had no impact on the formation of those conventions among the mathematical scientists of his day. Yet the adoption of this convention seems odder still in that it provides no particular advantage for the account of the rainbow. Conceiving the sun in its true position merely renders the ray HM effectively parallel to the horizon,²⁵ and so would require an alternative condition for determining the altitude of M, for the condition on the ratio of the lines HM, MK becomes meaningless. This

seems, then, to be another indication that the account of the rainbow has been fashioned to serve as the vehicle for this problem, rather than to provide the context for its original statement and solution.

If these considerations do not quite rule out the authenticity of *Meteorologica* Book III, they surely deserve more serious attention than Aristotle scholars tend to afford them. At the least, we recognize that the geometric locus problem in the passage on the rainbow is an importation from geometric practice, not Aristotle's own discovery. But the suggestion of its nonauthenticity at once raises the problem of dating. It has already been observed that the *Meteorologica* appears to be a composite document, different sections reflecting times ranging from the mid-350's to the late 340's.²⁶ Conceivably, then, some of its parts might be later still, and some might even be the work of disciples after Aristotle's death in 322. If so, how much later? If the passage on the locus problem derives from the 3rd century, near or even after the time of Apollonius, then its significance for the history of ancient geometry diminishes almost to nil.

In this instance we possess a valuable index of early provenance: the frequent appearances of archaic terminology in expressions like "the line *on which* (are) AB" (*hē grammē eph' hēi AB*), where standard usage after Euclid would read "the right line AB" (*hē eutheia AB*).²⁷ We have seen that the older form is characteristic of usage in the Eudemus fragment on Hippocrates cited by Simplicius, and an analogous form for logical terms is prevalent in Aristotle's *Prior Analytics*. By contrast, the shorter form is exclusive in Euclid, Archimedes, and Apollonius and the writers who followed them.²⁸ One must suppose that the author of the solution of the locus problem was in touch with the regular tradition of geometry of his time, so that if that happened to be much after the beginning of the 3rd century, the use of such nonstandard terminology would be difficult to explain.

Thus, I view the account of the rainbow in the *Meteorologica*, and the version of the locus problem to which it has been specially accommodated, as a product of the late 4th century, written more likely by a disciple of Aristotle than by the philosopher himself. The solution of that problem is not original in this context, as one sees from the absence of the analysis which would motivate the construction, but more clearly, from the partial garbling of the proof, needlessly encumbered by an indirect format of demonstration.²⁹ What, then, was the nature of the author's source and of the geometric activity which produced it? We have little means for determining this. But certainly when Apollonius reproduced this construction in his *Plane Loci* a century later, he was drawing not from the *Meteorologica*, but from its mathematical sources.³⁰ The *Meteorologica* passage thus reveals to us the manner in which this result was received by a mathematically oriented writer, late in the 4th century, and so indicates both the topicality and the significance of its discovery in the field of research then current.

EUCLID'S ANALYTIC WORKS

The problem from the *Meteorologica* manifests several features standard in the classical form of problem solving. It is a problem of locus, that is, the construction

of the set of points (typically a line or curve) satisfying a stated condition. But here it is actually expressed not as a problem, but as a theorem about "givens": that if two points are *given* and a ratio is *given*, then the circle is *given* which is such that the lines drawn from each of its points to the given points are in the given ratio. That Euclid devoted an entire treatise, the *Data* ("givens"), to theorems framed in this manner reveals that the form was important within geometric studies of the time. We recall, furthermore, that the Aristotelian problem assumes an auxiliary construction: that the line *Z* may be produced (*prospeporisthō*) such that for given lines *B*, *D*, $D : B = B + Z : D$. Here too Euclid composed a massive treatise of *Porisms*, now lost, so called perhaps only for their role as auxiliaries to other constructions, although the stronger sense of "production," that is, of terms of a stipulated description, might also apply.³¹ Most important, we have seen that the Aristotelian problem, while presented in strictly synthetic fashion, is best understood as framed in answer to a preliminary analysis. Clearly, this analytic method had attained prominence by Euclid's time, for Pappus includes these and other treatises by Euclid among the corpus of works, the *Topos analyomenos* (frequently translated as "Treasury of Analysis"), designed in the interests of analytic investigations.³²

The *Data* is a complement to the *Elements*, recast in a form more serviceable for the analysis of problems.³³ As in the Aristotelian example, each of its theorems demonstrates that a stated term will be given on the assumption that certain other terms are given. The subject matter overlaps that of the *Elements*, dealing with ratios and with configurations of lines and of plane figures, both rectilinear and circular. Indeed, only in rare instances does the *Data* present a result without a parallel in the *Elements*.³⁴ The proofs follow in a deductive sequence, so that one might refer back to previously proved theorems; in the Aristotelian problem, for instance, one assumes that the ratio of given terms is given, or that a term in given ratio to a given term is itself given—such propositions appear in the *Data* (Prop. 1, 2) and must have been available to the Aristotelian writer in a comparable form. Again, some proofs may assume results proved in the *Elements*; for instance, properties of similar triangles are needed in the Aristotelian problem. In view of this, one might suppose the *Data* was effectively superfluous in light of the *Elements*.³⁵

But the analysis we sketched for the locus problem in the preceding part reveals the special utility of the "data" format for this type of investigation. There, the analysis started from the supposition that the construction has been done, namely, that the point *M* is such that $HM : MK$ equals the given ratio. The construction of *P* then led to a set of ratios linking the lines *HK*, *KP*, *PM*. We next assumed an arbitrary line *D* as *given*; hence *B*, the line in given ratio to *D*, is also *given* (so that $D : B = PM : KP$). Furthermore, if *Z* is the line such that *D* is mean proportional between *B* and $B + Z$, then it too is *given*, and it has to *HK* the same ratio that *B*, *D* have to *KP*, *PM*, respectively. But *HK* is *given* (for the points *H*, *K* are given as basis of the locus to be constructed), so that the ratio $HK : Z$ is *given*; and *B*, *D* are *given*, so that *KP*, *PM* are *given* (the latter in magnitude only), whence the circle of center *P* and radius *PM* is *given*. Thus, the locus of *M* is *given*.

This ends the analysis and serves as the basis for the synthetic construction of the solution, as we meet it in the Aristotelian and Apollonian texts. One proceeds generally in the reverse order: starting from the given points H, K and the given ratio, one introduces auxiliary lines D, B in that ratio and produces the line Z, as described in the analysis; from this the lines KP, PM are determined and the circle is drawn. The proof consists of verifying that $HM : MK$ equals the given ratio. In this instance, each step of the analysis was simply convertible. But it sometimes happens that the reversal of the logical order is possible only when certain additional conditions are satisfied; the determination of the appropriate conditions would then require a separate treatment, called the "*diorismos*," in effect specifying when the problem as stated is in fact constructible.⁴⁰ The investigation of diorisms seems to have been advanced already by the middle of the 4th century, so that the procedure of analysis and synthesis was familiar as an instrument for the solution of problems several decades before Euclid.⁴¹

Referring back to our sample analysis, one observes how the concluding section is shaped by a series of "givens," each determined from preceding ones, until one has arrived at the desideratum of the construction as given. The adoption of this terminology is not merely a formalism; it serves a critical purpose by keeping separate two sets of terms having quite different logical status. Some terms are known only conditionally, in that they follow upon the hypothesis of the analysis that the construction has been effected; but other terms are known by virtue of the definition of the construction which one is to produce. The latter are the "givens." When these two sequences of terms arrive at a common term, one may consider the analysis to be complete. Without a terminological ploy like that of the "givens," the logical status of these sequences would soon become hopelessly confused. Thus, far from being superfluous, Euclid's *Data* accords formal recognition to the advanced stage of analytic researches at his time.

The *Meteorologica* passage already employs the form of sequential "givens," so that Euclid cannot be taken as having originated this style, although it is possible that his *Data* represents the first major effort to organize the materials of elementary geometry in this form. Indeed, some parts of the *Data* are modeled after their analogues in the *Elements*, indicating that these parts are not likely to have had an independent provenance.⁴² We lack earlier instances of the sequential usage, doubtless owing to the general loss, but for fragments, of documentation from the pre-Euclidean period, so that one cannot specify the origin of this usage. But for the simple notion of the "given" in geometry, one has evidence from early in the 4th century in the "hypothesis" passage from the *Meno* (86e-87b), where the project is to apply a certain area "along the given line of this circle here. . . ." The phrasing is somewhat loose, as the dialogue context would recommend, but the formal basis can be seen: the conditions to be satisfied in the construction of a problem are framed in terms of certain "given" magnitudes. This is invariably the form in the problems in Euclid's *Elements*.⁴³ It is perhaps best to view this as a natural development, formalizing usage in common language. For one comes upon comparable expressions in

dialectical passages, for instance, without geometric connotations. Aristotle describes the good legislator as "capable of taking the *given* constitution into account," that is, the one actually existing, rather than some hypothetical ideal (*Politics* IV, 1, 1288 b 28); or, again, he dismisses the notion that an eternal thing might be subject to generation or passing away, as this "conflicts with one of the *givens*," that is, the generally accepted principles of natural philosophy.⁴⁴ In view of such parallels, however, one should not rule out the possibility that geometry and dialectic interacted in the development of this usage, at a time when both fields were seeking to formalize the elements of research.

A further contribution by Euclid to the study of advanced problems lies in the field of the conic sections. In the production of his treatise *On the Conics*, Euclid was able to draw on the efforts of an earlier contemporary, named Aristaeus "the elder" by Pappus, for he is said also to have written a treatise on this subject.⁴⁵ As neither work survives, doubtless owing to their having been superseded by Apollonius' far more extensive treatment about a century later, our knowledge of their nature is indirect and severely limited. One must suppose that whenever Archimedes assumes without proof a theorem on conics, "for that has been demonstrated in the *Conic Elements*," he refers to the work either of Aristaeus or of Euclid.⁴⁶ From Apollonius himself and commentary by Pappus and Eutocius, we learn that most of the materials on the general properties of the curves in Apollonius' Books I and II were familiar within the older theory, and that his third book extended a body of theorems useful for the solution of locus problems initiated by Euclid. As for the fourth book, investigating the manner of intersections of conics, Apollonius assigns its origins to Conon of Samos, a geometer in the generation just after Euclid and an important influence on Archimedes' early work. The remaining four books on normals, similar conics, and conjugate diameters were the product of Apollonius' own more advanced researches.⁴⁷ Thus, the theory at Euclid's time may be viewed as consisting essentially of the materials of the first two books and some portions of the third.⁴⁸

As we have seen, the standard view that the theory of the conics was initiated and strongly advanced by Menaechmus around the middle of the 4th century is open to serious question. His treatment of the cube-duplication problem indicates an interest in the family of curves specified via second-order relations in accordance with the techniques of the application of areas, and he and his followers surely were able to discover properties of such curves and use them in the solution of other problems.⁴⁹ But the original context of the discovery that such curves resulted from the sectioning of cones is a matter of conjecture. O. Neugebauer, for instance, has suggested the study of the shadows cast in sundials as a possible context.⁵⁰ Alternatively, the field of optics seems a likely prospect. For instance, Euclid demonstrates that "the wheels of chariots appear sometimes circular, sometimes *drawn in (parespasmenoi)*" (*Optics*, Prop. 36), and Pappus includes among his lemmas to the *Optics* the construction of the ellipse whose appearance the circle assumes in the latter case.⁵¹ Moreover, the earliest extant references to the conic section have to do with the same situation: for instance, that the

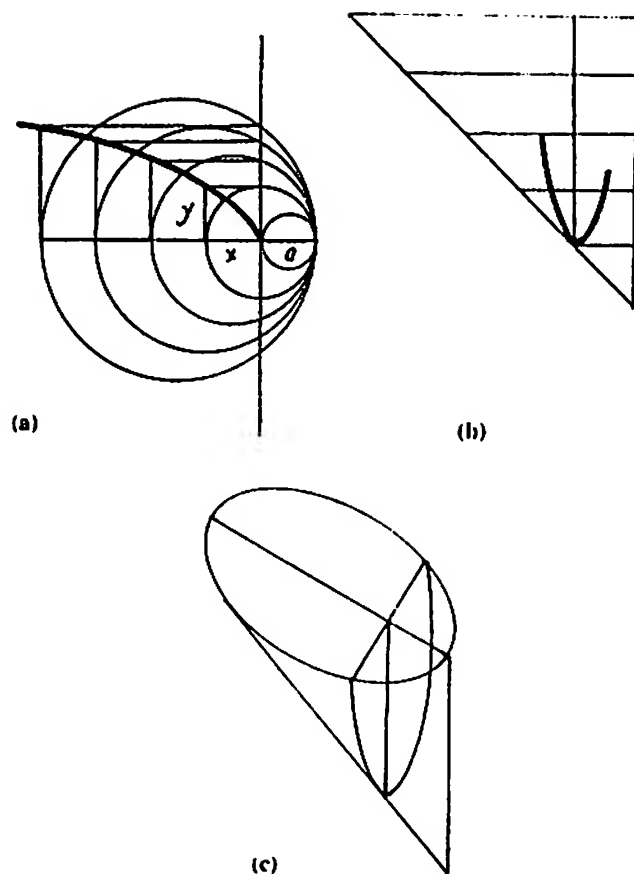


Figure 5

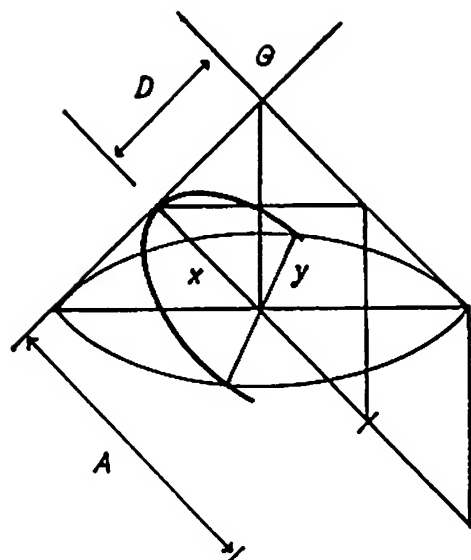
gibbous moon looks like a "shield" (*thyreas*) or that a circle viewed obliquely takes the shape of "the section of an acute-angled cone."¹⁸

Even if my view of the earliest phase of the study is correct, we should still recognize that the Greek geometers could not long have remained content with the type of pointwise constructions we have proposed. Curves arose as abstractions of motions (e.g., circles and spirals) or through the intersection of solids (e.g., the curve of Archytas). The sort of algebraic relation by which one defines a curve in the Cartesian manner was admitted by the Greek as a fundamental *property* (*symptōma*) of the curve, but not as the basis of its conceptualization.¹⁹ One can imagine how the search for a natural mode of generation might have progressed from the pointwise construction to the sectioning of cones. For instance, the curve introduced via the relation $Y^2 = AX$ might entail for its pointwise construction a diagram consisting of a system of circles sharing a vertex and having their diameter lying along the same line [Fig. 5; cf. Ch. 3, Fig. 12(c)]; for each circle the chord is drawn perpendicular to that diameter, as to

divide the diameter into segments equal to A and X , so that the chord itself will be twice Y .²⁰ One might then see, if the lines X are drawn in elevation at right angles to the plane, that the diagram becomes the plane projection of a cone, one of its generators being the line perpendicular to the plane and passing through the common vertex of the circles, and the curve being formed as the cone's intersection with a plane parallel to that generator.

Under this view, the study of these curves, as defined via the application of areas, had progressed significantly before it was discovered that the curves formed as sections of the cone satisfy the same second-order relations. The work of producing a full-fledged treatise on the conic curves, like those by Aristaeus and Euclid, could proceed quite rapidly, for most of the properties of these curves are deduced from their *symptōmata*, the second-order relations, rather than from their generation as sections. It is important to realize that the role of the latter aspect of the curves was to provide a natural conceptual view of their generation. It did not serve for their practical construction, for later writers found pointwise procedures far more convenient than any solid construction.²¹ On the other hand, it was largely superfluous for the work of proving theorems, at least in the more advanced stage of the theory at the hands of Apollonius. Realizing this helps to explain another rather puzzling feature of the early theory of the conics. We learn from Pappus and Eutocius that before Apollonius the conics were specified such that the sectioning plane was taken perpendicular to the side of the cone.²² Thus, parabolas were formed only via the perpendicular sectioning of the right cone with a right angle at its vertex, and were accordingly named "sections of the right-angled cone." Similarly, ellipses and hyperbolas were formed, respectively, from cones with acute and obtuse angles at the vertices. The report of the commentators is confirmed by usage in Archimedes and Diocles, and the older names persist in some accounts well after Apollonius.²³ Yet why this restriction? It seems to belie the fact that specimens of all three types can be formed as sections from any cone—a consideration which surely must have led Apollonius to remove this restriction on sections, although the nature of oblique sections was already well appreciated by Euclid and Archimedes before him. Odder still, it prohibits the formation of a circle as a "conic section." Clearly, then, the retention of this mode for so long was due to its introduction as a deliberately artificial aspect of the theory; that is, it reflected not the primitiveness, but the sophistication of that theory around the time of Euclid.

Our account of the initial phase of these studies revealed that the second-order curves at first lacked a natural mode for their generation. The sectioning of the cone could provide a suitable mode, and indeed under the restricted manner of sectioning there will be a unique section corresponding to each second-order curve.²⁴ To see this, first consider the parabola: in the application mode it is specified via a single parameter, the line segment over which the application is performed (line A in the relation $Y^2 = AX$). Correspondingly, when it is formed from the perpendicular section of a right-angled cone, a single term, the distance D of the cutting plane from the vertex of the cone, will designate the curve. [See Fig. 6(a).] It follows that $A = 2D$. Since the distance along the diameter



(a)

Figure 6

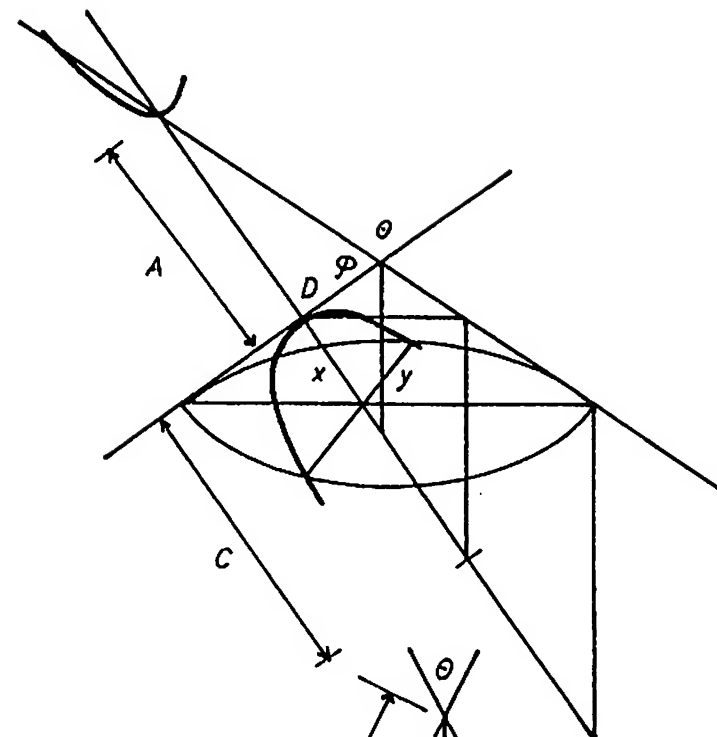
of the section from its vertex to where it meets the axis of the cone also equals D , we obtain a straightforward sense for Archimedes' expression of the parameter as "double the line up to the axis."⁵⁵ Turning now to the hyperbola and the ellipse, we may specify each form via two lines such that an area similar to the rectangle defined by the two lines is applied to one of those lines as to exceed or fall short by a square (i.e., the lines A , C in the relations

$$\frac{A}{C} Y \cdot Y = AX \pm X^2$$

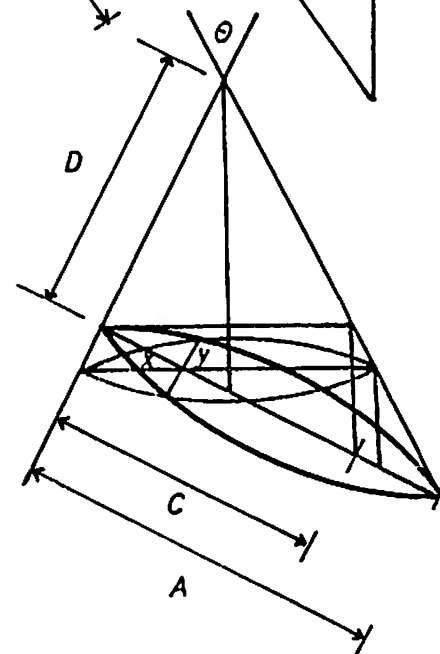
where the positive sign corresponds to a hyperbola, the negative sign to an ellipse). We have two important indices supporting this view of the initial forms of the curves. First, it leads immediately to the form implied as the standard in Archimedes' theorems, namely as $Y^2 : X (A \pm X) = C : A$.⁵⁶ Second, Pappus' statement of the definitions explicitly figures the excess or defect as a *square*. But in Apollonius' definitions there is a subtle difference; for the "latus rectum" C and "diameter" A give rise to the relation

$$Y^2 = CX \pm \frac{C}{A} X \cdot X$$

so that the excess or defect is similar to the figure defined by the lines C , A . Pappus has not erred, as some have charged; he has merely followed a source which conforms with the pre-Apollonian designations.⁵⁷ Like the parabola, each hyperbola and ellipse may be associated uniquely with a perpendicular section of a cone [see Fig. 6(b) and 6(c)], where the distance D between the vertex of the section and the vertex of the cone and the angle θ at the vertex of the cone



(b)



(c)

are determined from the lines A , C via the relations $C = 2D \tan (\frac{1}{2} \theta)$ and $A = D \tan \theta$ (for the ellipse, θ acute) or $A = D \tan \phi$ (for the hyperbola, θ obtuse and ϕ its supplement).⁵⁸ Note that just as for the parabola, the latus rectum may be styled as "double the line up to the axis."

This effort to relate the second-order relations to the sections of the cone produced constructions to form an appropriate conceptual background for their study. But it would not much affect the content or order of the theory, or its primary purpose, to serve as an instrument for the investigation of problems. From the start, it was in their form as area relations that the curves had their application, and this remained so throughout the later development of the theory. We have seen how the cube duplication figured prominently in the initial phases of this work, and we shall see in the angle trisection another significant example of its use. Apollonius informs us that Euclid made a start in another area where the conics are indispensable: the study of the problem of the locus of three and four lines.⁵⁹ In this, one seeks a construction of the set of points such that given three (or four) lines, the distances A, B, C (and D) of each point to the respective given lines satisfy the condition that the ratio $A \cdot B : C^2$ (or $A \cdot B : C \cdot D$) is given. In general, the locus is a conic section. Theorems from Apollonius' third book (Prop. 16–22), on the ratios of segments of tangents and of intersecting chords, have their application in the solution of this problem, as Apollonius observes, noting in strong terms the inadequacy of Euclid's version: "he did not effect the synthesis [of this locus] save for the chance part of it and that not successfully," for only with the aid of Apollonius' own newly discovered theorems could that be done. Zeuthen has suggested that Euclid would have been hindered in this effort through his failure to conceive of the two branches of the hyperbola as constituting a single curve.⁶⁰ But it seems that Apollonius levels a more serious charge than this would indicate. Unfortunately, understanding this issue depends heavily on reconstructions, for neither the Euclidean nor the Apollonian investigation of this locus survives. But it can be hoped that insight into the former may follow from consideration of another of the Euclidean works Pappus includes in the analytic corpus, the *Porisms*.

The three books of Euclid's *Porisms*, according to Pappus, formed a massive treatise of 171 theorems falling into 29 classes, yet without claim of exhausting the subject matter.⁶¹ Pappus gives a paraphrase collecting into a single proposition several of Euclid's; the first part, incorporating ten propositions of the first species, is this:

if three points on one line of the *hyption* or *parhyption* figure are given, and the others save for one lie on a line given in position, then that also lies on a line given in position.

That is, with respect to the two configurations, the "supine" figure [Fig. 7(a)] and the "hypersupine" figures [Figs. 7(b) and 7(c)], if points A, B, F are given and points C, D lie on given lines, then E also lies on a given line.⁶² One recognizes the connection with those studies now included within the field of projective geometry and which received their modern treatment through researches initiated in the 17th century by Desargues and Pascal.⁶³ For instance, the converse of the lemma just cited from Pappus provides a proof of the theorem of Desargues on two projectively related triangles: that if the lines joining corresponding vertices of the triangles CED, C'E'D' meet at a point, then the points of intersection A, B, F of corresponding sides are collinear [see Fig. 7(d)].⁶⁴ A

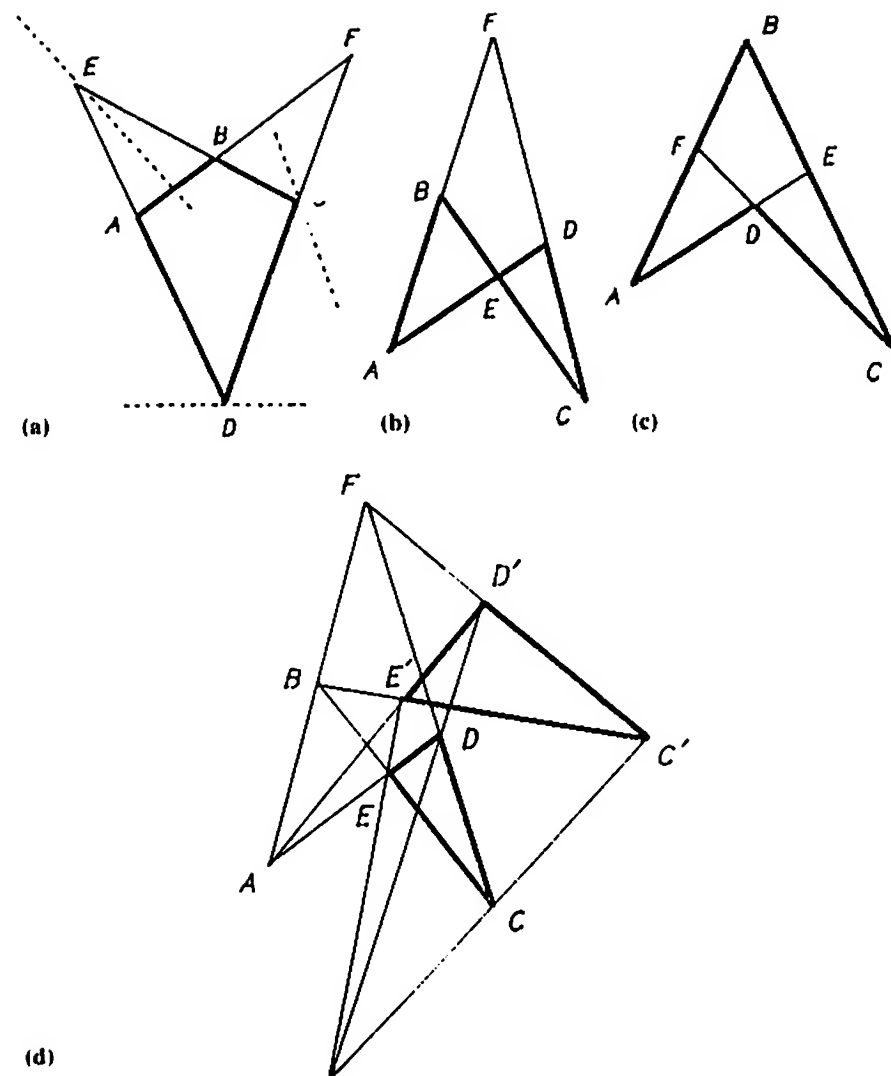


Figure 7

perusal of the 38 lemmas which Pappus presents on the *Porisms* confirms this sense of the affinities between the subject matter of the *Porisms* and that of the more modern field of projective geometry. In addition to the connection with Desargues' theorem just cited, one finds configurations relating to the complete quadrilateral, the preservation of cross ratio through projective transformations, and results on involutions. Most noteworthy is the famous theorem of Pappus on the hexagon inscribed between two lines:

if AB, GD are parallel lines and there fall on them certain lines AD, AZ, BG, BZ and ED, EG are joined, then a straight line arises through H, M, K.⁶⁵

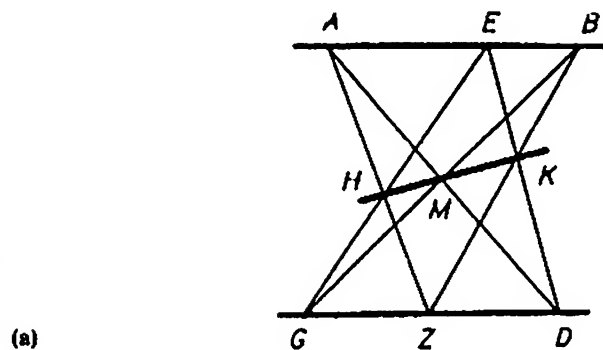
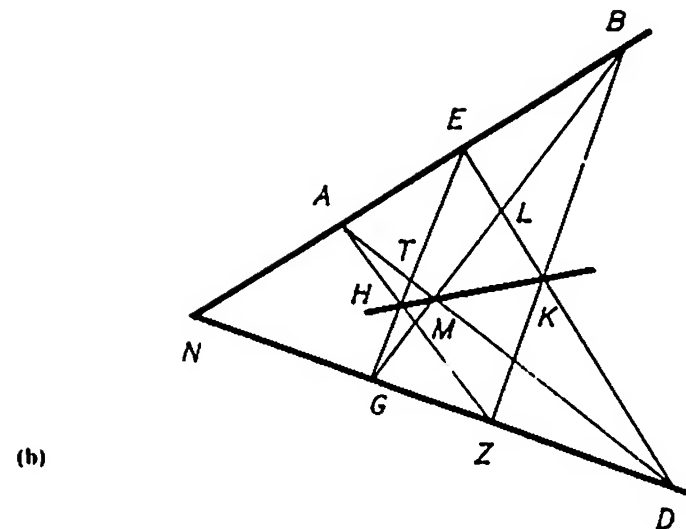


Figure 8

[See Fig. 8(a).] With such extensive information to guide them, several scholars have attempted reconstructions of this lost work. The most ambitious effort is that of Chasles, who perceived a clear link between the ancient study and his own researches in projective geometry.⁶⁶ Working through the many cases suggested by configurations such as these, Chasles compiled a series of over 200 propositions whose demonstrations could be managed within the technical domain available to Pappus. However impressive this project might seem, others appear to have been disappointed. For surely Pappus' lemmas are but a mere shadow of the scope of Euclid's work itself.⁶⁷ Thus, Zeuthen refers to the more general context of these studies in the modern theory—for instance, the theorem of Pappus is a special case of Pascal's theorem on hexagons inscribed in conics—and so proposes that Euclid was moving toward a projective theory of the conic sections.⁶⁸

It would appear, however, to be a questionable procedure on our part to extrapolate in this way and insist that because the *potential* for such a theory is foreshadowed in these lemmas, the work to which they relate, Euclid's *Porisms*, must *actually* have advanced such a theory. It is an interpretive issue of no small subtlety to determine in what sense and to what degree Euclid and later geometers developed a certain body of geometric materials which have been subsumed in more recent times within the field of projective geometry. This caution is especially well advised because the ancient manner of effecting these results entirely lacks the conceptions and methods characteristic of the modern field. Consider, for instance, Pappus' treatment of the theorem on the inscribed hexagon. He establishes this in two cases, as the bounding lines are parallel (Lemma 12) and as they intersect (Lemma 13)⁶⁹; in this he already betrays the absence of an essential feature of projective methods: the equivalence of parallel and nonparallel configurations through the introduction of points and lines at infinity. In the nonparallel case, with reference to the lines GHTE and NGZD, Pappus has, from a prior lemma (3), $GE \cdot HT : GH \cdot TE = GN \cdot ZD : ND \cdot GZ$; similarly, with reference to NGZD and DKL E, he has $GN \cdot ZD : ND \cdot GZ = DK \cdot EL : DE \cdot KL$ [see Fig. 8(b)].⁷⁰ It thus follows that $GE \cdot HT : GH \cdot TE = DK \cdot EL :$



(b)

$DE \cdot KL$. He can thus conclude from another lemma (10) that HMK is a straight line. Pappus' proof is structurally identical to the standard projective one: the perspectivity centered on A projects GHTE onto GZDN, while that centered on B projects GZDN onto LKDE; the projection GHTE onto LKDE is the composition of these two perspectivities, and since it has the fixed point E, it is itself a perspectivity. The center of projection must be the intersection of GL, TD, namely, the point M; hence the line HK passes through M.⁷¹ From a comparison of these two treatments, one sees that Pappus' Lemma 3 amounts to the specification of a projective transformation (here, a perspectivity) via its cross-ratio-preserving property; its converse (Lemma 10) plays the role which the modern approach casts to the fundamental theorems on the composition and determination of projectivities and perspectivities.

There emerges from this a tantalizing puzzle: in view of the utter absence of projective techniques from the ancient treatment, how did it happen that the author of the *Porisms* came to investigate the same configurations and properties which are prominent in the modern theory? Just what was Euclid up to? What we know of this work indicates that it was ancillary to a larger field of research.⁷² Indeed, Pappus says so explicitly: "the *Porisms* of Euclid are a most ingenious set mustered toward the analysis of the more heavy-laden problems," and notes their utility for locus problems of the sort found in abundance in the *topos analyomenos*.⁷³ Now, as we have noted, results comparable to some in the *Porisms* have been applied with stunning success toward the study of conics since the 17th century. In the light of Euclid's contributions to the early theory of conics, one may well conjecture that Euclid compiled the *Porisms* to support a problem-solving activity related to the conics. This view would be a somewhat weakened form of that advocated by Zeuthen. Let us then consider one of the problems of this sort expressly assigned to Euclid, the locus with respect to three

and four lines, to see whether it can provide insight into the nature and the role of the *Porisms*.

THE ANALYSIS OF CONIC PROBLEMS: SOME RECONSTRUCTIONS

One may observe that the pre-Apollonian forms for the central conics express them as special cases of the three-line locus. In the defining relations $Y^2 : X \cdot (PP' \pm X) = L : PP'$, for given lines L , PP' , the lengths X and $PP' \pm X$ represent the distances of each point of the conic from two given parallel lines, as measured in the direction of a given transversal line, while Y represents its distance from that transversal, as measured in the direction of the parallels (see Fig. 9). One thus sees that the ellipse (for $PP' - X$) and the hyperbola (for $PP' + X$) which are tangent to the parallel lines at P , P' and have PP' as diameter and L as latus rectum will satisfy the property of the three-line locus relative to these given lines and the given ratio $L : PP'$. In effect, then, the three-line locus problem seeks to generalize the defining condition of the conics for other configurations of the reference lines.

Solutions of the general problem can be constructed on the basis of two propositions from Apollonius' *Conics*:

III. 16: if OP , OQ are tangent to a conic and KK' is a secant parallel to OQ which meets the conic in R , then $OP^2 : OQ^2 = PK^2 : KR \cdot RK'$ [see Fig. 10(a)];

III. 17: if RR' , SS' are secants parallel to OP , OQ , respectively, and meet each other in J , then $OP^2 : OQ^2 = RJ \cdot JR' : SJ \cdot JS'$. [See Fig. 10(b)]¹⁴

The latter theorem holds not only when J is an internal point (as shown in Fig. 10b), but also when the secants intersect externally. The theorems are related

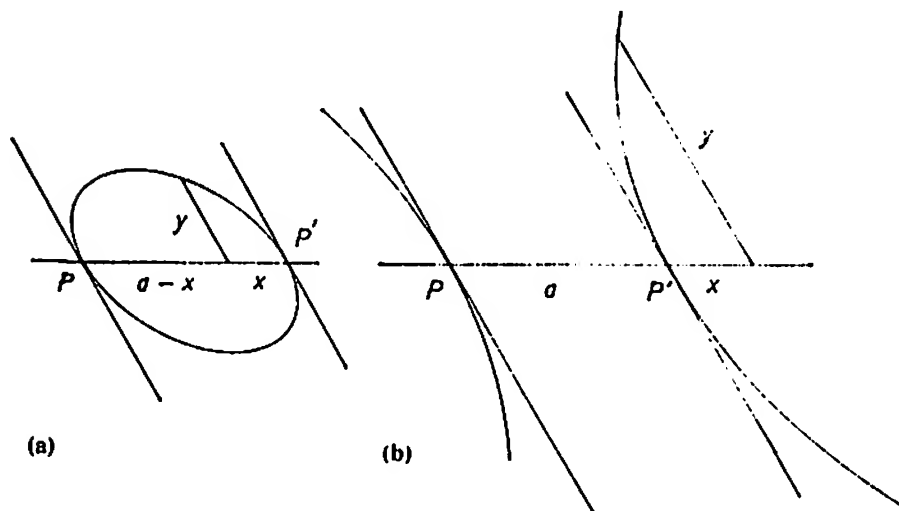


Figure 9

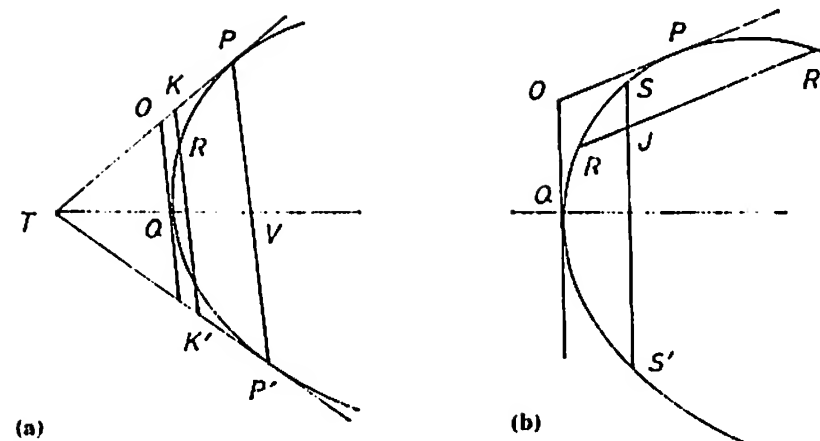


Figure 10

in that they each generalize familiar properties of tangents and secants to the circle, where $OP = OQ$, $PK^2 = KR \cdot RK'$ and $RJ \cdot JR' = SJ \cdot JS'$ (cf. Euclid's *Elements* III, 35, 36). Since Archimedes asserts as "proved in the *Conic Elements*" this same property of the intersecting chords in a conic (*Conoids and Spheroids*, Prop. 3),¹⁵ we thus perceive that this property was established within the earlier theory of conics.

The fact that a conic has the property of the three-line locus follows at once from III. 16,¹⁶ for if we draw tangents TP , TP' to the conic, bisect chord PP' in V , and join TV meeting the conic in Q , then TV will be a diameter corresponding to the ordinates parallel to PP' [see Fig. 11(a)].¹⁷ Moreover, if OQ is

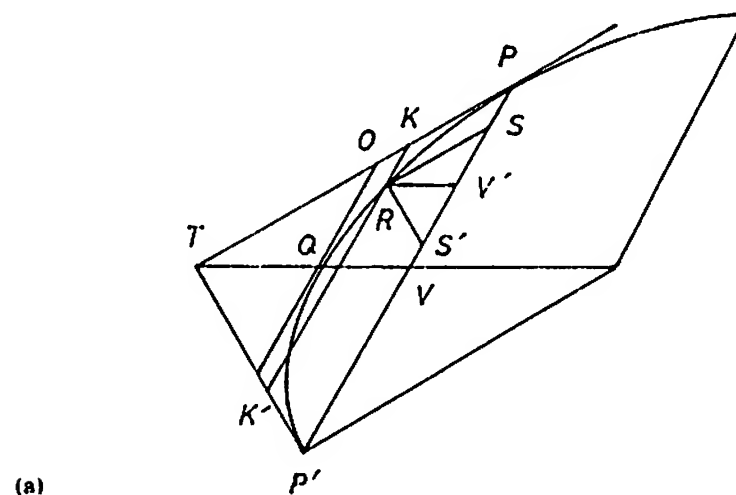
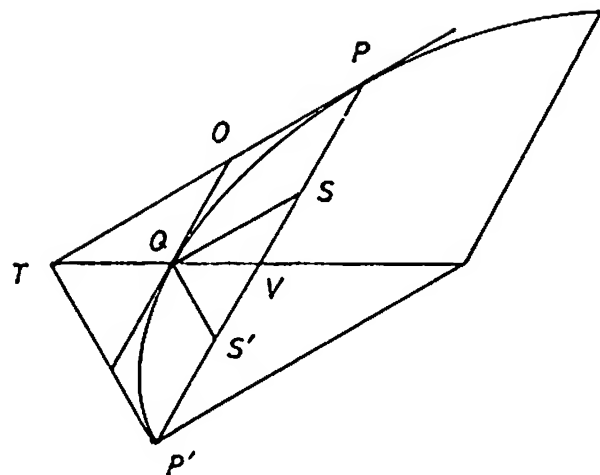


Figure 11

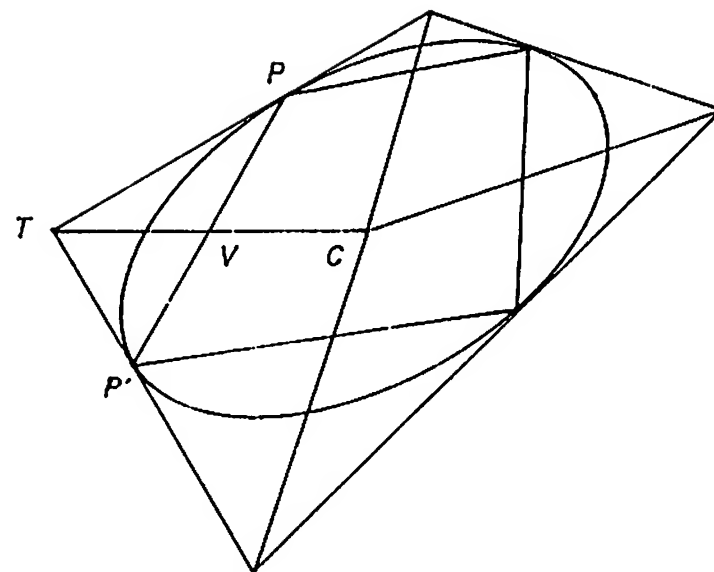


(b)

drawn parallel to PP' , it will be tangent to the conic at Q . If next we draw lines from R parallel to TP , TV , TP' meeting PP' , respectively, in S , V' , S' , then $KR = PS$, $RK' = S'P'$ and $PK : RV' = PT : TV$. We thus have $OP^2 : OQ^2 = PK^2 : KR \cdot RK' = (RV'^2 : PS \cdot S'P') \cdot (PT^2 : TV^2)$, so that $RV'^2 : PS \cdot S'P'$ has a given value, independent of the choice of point R on the conic. As the lengths PS , $S'P'$, RV' are the respective distances of R from the lines TP , TP' , PP' as measured in the given directions of PP' , TV , one sees that R lies on the three-line locus relative to the lines TP , TP' , PP' , and the given value of the ratio.

Conversely, to find the curve answering the condition of the three-line locus, we note that if Q is the intersection of the curve with TV , then $QV^2 : PS \cdot S'P' = M^2 : N^2$, a given ratio [see Fig. 11(b)]. Since $PS = S'P' = OQ$, and $OQ : TQ = PV : TV$, one has $QV : TQ = (M : N) (PV : TV)$, a given ratio. We may thus determine Q on the given line VT by means of this proportion and then introduce the conic of vertex Q , diameter TV and tangents TP , TP' . That is, when Q bisects VT , the conic is the parabola whose latus rectum L satisfies the relation $PV^2 = QV \cdot L$. Otherwise, we find Q' via the harmonic relation $QV : VQ' = QT : TQ'$; then QQ' will be the diameter of the conic and L its latus rectum, for $PV^2 : QV \cdot VQ' = L : QQ'$. One notes that this yields an ellipse when QV is less than TQ and a hyperbola when it is greater.

In this way, the solution of the three-line problem, where the solving locus is a parabola, an ellipse, or a single branch of a hyperbola, was available to anyone knowing the property proved in *Conics* III, 16. An aspect of the problem not within the range of the theory of conics before Apollonius relates to the cases where both branches of the hyperbola are considered.⁷⁸ We may recognize two additional configurations analogous to III, 16: one in which the tangents are drawn to P , P' on one branch and R lies on the second; another in which the tangents are drawn to different branches. Apollonius presents the analogues in III, 18, 19 from which one can complete the construction of the locus.



(c)

Having established that the conic satisfies the property of the three-line locus, one can derive that it satisfies the property of the four-line locus by considering the tangents drawn to the four vertices of an inscribed quadrilateral and applying the three-line result in succession to each of the four sides [see Fig. 11(c)].⁷⁹ But the conversion to a construction of the four-line locus is not as straightforward as that of the three-line locus. Zeuthen has thus proposed a different construction developing from the relation of the segments of intersecting chords (III, 17).⁸⁰ In the first instance, let the given four lines intersect in the trapezium $ABCD$, sides AD and BC being parallel; we wish then to determine the locus of points R such that $RI \cdot RK : AI \cdot IB$ has a given value, the four segments here being the distances from R to the respective sides of $ABCD$ as measured in the directions AD , AB (see Fig. 12). Let us conceive the conic curve about $ABCD$ whose diameter is the bisector of the parallel sides AD , BC and whose ordinates lie in the direction of AD . Then by III, 17, the ratio $RI \cdot IS : AI \cdot IB$ has a constant value for all chords drawn parallel to fixed directions, while $RK = IS$. Hence, this conic will be a solution for the locus of R if it can be contrived that this ratio has the value specified in the locus problem when the chords are parallel to AD , AB , respectively. To construct this conic, Zeuthen proposes the introduction of an auxiliary conic which will turn out to be similar to that required. Since the tangents parallel to AD , AB meet in segments such that the ratio of their squares $OP^2 : OQ^2$ equals that of the products of the segments of the intersecting chords, we may assign segments $O'P'$, $O'Q'$ respectively parallel to AD , AB , such that $O'P'^2 : O'Q'^2$ has the given value, and draw $P'T'$ parallel to the bisector of AD , BC . In this way, $O'P'$, $O'Q'$ will be tangents and $P'T'$ the direction of the diameter of the auxiliary conic similar to the one sought. Extend

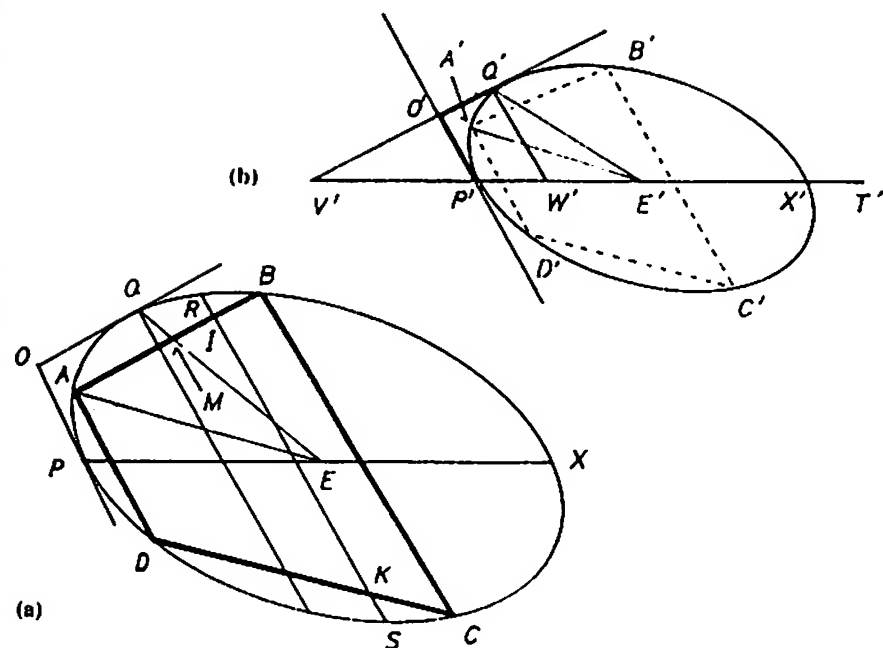


Figure 12

$O'Q'$ to meet $P'T'$ at V' , and draw from Q' a line parallel to $O'P'$ meeting $P'T'$ at W' . We now produce the point X' on $P'T'$ in harmonic relation with V' , P' , W' ; that is, $P'V' : P'W' = X'V' : X'W'$. Thus, $P'X'$ is the diameter of the conic (via *Conics* I, 36) and its latus rectum L' is determined from the relation $Q'W'^2 : P'W' \cdot W'X' = L' : P'X'$ (via I, 21). To complete the construction, one must circumscribe about $ABCD$ the conic similar to that in the auxiliary figure. For instance, since the bisector E' of $P'X'$ is the center of the auxiliary conic, the direction $E'Q'$ is known; the analogous line in the required figure must meet the diameter at the same angle and pass through the midpoint M of AB , a chord parallel to the tangent corresponding to OQ . This determines the center E of the required conic. If we join E, A , the direction of the corresponding line in the auxiliary conic is the same, whence A' is found as the correlate of A . We can now inscribe the trapezium $A'B'C'D'$, similar to $ABCD$, and from it determine the points O, P, Q, X which specify the conic. Alternatively, once the ratio $EQ : EP$ of conjugate diameters has been found, Zeuthen proposes an application of *Conics* III, 27 from which the diameter of the required conic can be determined, thence the conic itself.⁸¹

In Zeuthen's account, not only could Euclid produce a solution of this sort for the case of the four-line locus where the reference figure is a trapezium, but also he could extend this to the general quadrilateral. For this Zeuthen uses a lemma based on one of the locus propositions Pappus cites from the *Porisms*.⁸²

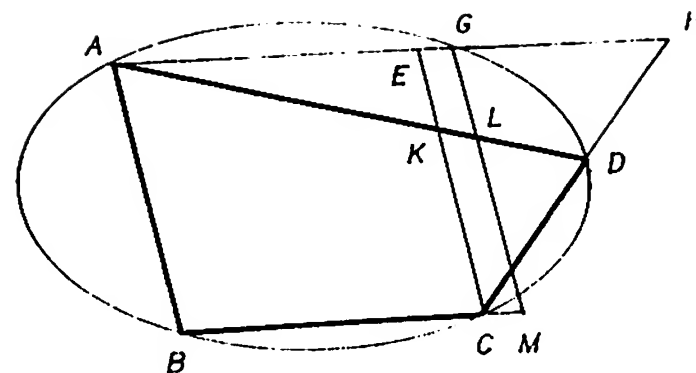


Figure 13

Unfortunately, Zeuthen must assume that Euclid had access to an extension of the cited proposition in which the locus of a given point can be taken to be not only a line, but also a conic (that is, a curve described by the four-line locus). But one may also see that Zeuthen's method is unnecessarily complicated; for the transition from the trapezium to the general quadrilateral can be effected without recourse to the lemma he employs. Let the quadrilateral $ABCD$ be the reference figure for a four-line locus, such that distances are to be measured in directions parallel to AB, BC (see Fig. 13). We complete the parallelogram $ABCE$ and extend AE to meet the solution curve of the locus in G ; let AG meet CD in H and let AD meet EC in K ; and let the line from G parallel to EC meet AD in L and BC in M . Since G is on the locus, $GH \cdot GA : GL \cdot GM = r : s$, the given ratio; while $GM = EC$ and $GA : GL = AE : EK$ via similar triangles. This determines GH , so that the given figure $ABCD$ determines G on the locus. Since the five points A, B, C, D, G determine the conic uniquely, one is thus permitted to solve the locus via the trapezium $ABCE$.⁸³

As in the case of the three-line locus, the complete solution of the four-line locus would be out of reach before Apollonius' introduction of the two-branched conception of the hyperbola. Securing the requisite lemmas is the project in III, 20-23. One would thus suppose that only the partial solution, perhaps along the lines of the method reconstructed by Zeuthen, was worked out by Euclid and his followers, then later extended by Apollonius.

A puzzle attaches to the set of theorems which close Apollonius' Book III. Like the propositions (16-23) just considered, the later set (Prop. 53-56) deal with relations of segments of secants and tangents drawn to conics. Consider III, 54 (see Fig. 14): if TQ, TQ' are tangents to a conic, V the bisector of chord QQ' , and if to any point R on the curve lines are drawn from Q and Q' to meet in S' and S , respectively, the line drawn from Q' parallel to TQ and the line from Q parallel to TQ' , then the segments $QS, Q'S'$ contain a given area; that is, $QS \cdot Q'S' : QQ'^2 = (PV^2 : PT^2) (TQ \cdot TQ' : QV^2)$, a given ratio.⁸⁴ Zeuthen suggests that this theorem might play a role in Apollonius' own solution of the

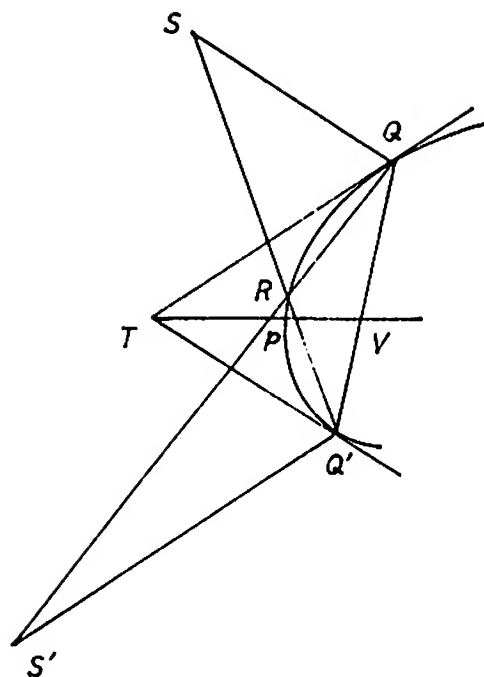


Figure 14

three-line locus problem, and this has been accepted by Heath and others.⁸⁴ But this view can hardly be correct, since III, 54 is merely an elaboration of III, 16, while the latter itself suffices for a straightforward solution of this problem, as we have seen. Elsewhere, Zeuthen notes the patent projective character of III, 54: for the constancy of the product $QS \cdot Q'S'$ enables one to assign to each line in the pencil at Q a unique line from the pencil at Q' , and this correspondence specifies a conic projectivity in the manner of Steiner.⁸⁵ To be sure, this element of the theorem is implicit in it. But even if one can now perceive here the "germ of the projective generation of the conic sections",⁸⁷ one can well doubt whether any ancient geometer would have conceived the same result in this sense.⁸⁸ Indeed, Apollonius' theorem immediately suggests his intent to solve a specific problem: to determine a conic passing through three given points and being tangent at two of these to given lines. For the givens specify the value of $QS \cdot Q'S' : QQ'^2$, so that the ratio $PV : PT$ is given from III, 54. This determines P and also the harmonic conjugate point P' (from $PV : PT = P'V : P'T$), so that the diameter PP' is known. From the relation $QV^2 : PV \cdot VP' = L : PP'$, the latus rectum L is also known. We have thus found the required conic.⁸⁹ As before, the other propositions (55 and 56) permit solutions for other configurations of the problem, where both branches of a hyperbola are to be considered. In view of this straightforward interpretation of these theorems, Zeuthen's projective hypothesis becomes less persuasive.

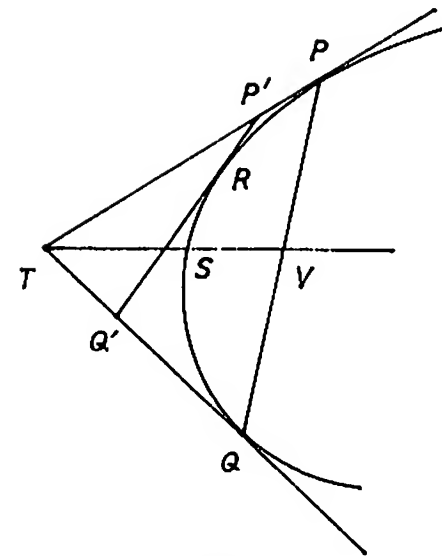


Figure 15

An interest in constructing conics in answer to such initial conditions of incidence and tangency may be detected in other parts of the *Conics*. For instance, one learns in III, 41 that three lines tangent to a parabola cut each other in proportion: that is, $PP' : P'T = P'R : RQ' = TQ' : Q'Q$ (see Fig. 15). The corresponding problem would be to determine a parabola tangent to three given lines and passing through a given point on one of them. Knowing any one of the points P , Q , R , the other two can be found via the relation in III, 41. If V bisects PQ , then TV will be the diameter of the parabola, and if S bisects TV , S will be its vertex (for PT is tangent, so $VS = ST$); thus $PV^2 = SV \cdot L$ determines its latus rectum L . In our discussion of Apollonius in Chapter 7, we shall find that the detailed examination of problems of this sort formed a major area of interest within his work.⁹⁰

Euclid's efforts on problems like that of the three- and four-line locus thus initiated a fruitful activity of problem solving within the range of the developing theory of conics. The fragmentary character of the extant evidence limits one's ability to distinguish Euclid's own findings and methods from those of the later geometers who continued his work. But our brief consideration here of samples from this field should make an important aspect of this work clear: that its concerns and methods were decidedly *not* projective. Zeuthen's perception of correspondences with the objectives of modern projective geometry seems rather to mistake technical coincidences for essential insights into motivation. Our hope to find within this material the context providing the rationale for the Euclidean *Porisms* has thus been frustrated; for it is the markedly projective appearance of its subject matter which most requires explication. The key we require will emerge, as we consider yet another of the lost Euclidean treatises, the *Surface Loci*.

AN ANGLE TRISECTION VIA "SURFACE LOCUS"

Pappus presents a series of three angle trisections, of which the third raises certain suggestions of an origin near the time of Euclid. We may sketch it as follows (see Fig. 16):⁹¹

Let it be required to cut the given arc ABG at B such that arc BG is its third. (Analysis:) let it be done, so that angle BGZ is twice BAZ . Draw GD to bisect angle BGZ , meeting BA at D , and draw DE and BZ perpendicular to AG . Thus, $AD = DG$ and $AE = EG$. Since angle BGA is bisected, $AG : BG = AD : DB$ (*Elements* VI, 3); but $AD : DB = AE : EZ$ (via similar triangles), so that $AG : AE = BG : EZ = 2 : 1$. Thus, B lies on the locus such that $BG = 2EZ$, for given points G, E ; that is, the locus which is such that $BZ^2 + ZG^2 : EZ^2 = 4 : 1$. This is a hyperbola, given in position, so that its intersection with the given arc ABG solves the problem. And the synthesis is obvious.

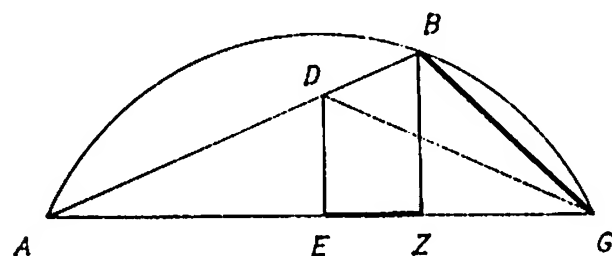


Figure 16

One sees that the angle trisection has been reduced to the specification of a hyperbola via focus and directrix, that is, as the locus of points whose distances from a given point and a given line are in a given ratio greater than $1 : 1$. But we note that the locus is first transformed into one involving the *squares* of those distances, rather than the distances themselves. Elsewhere, Pappus presents a proof of this locus construction, and there too the problem is expressed in terms of the squares of the distances.⁹² It is part of a general account of the focus-directrix specification of all three conics which Pappus provides as lemmas to Euclid's *Loci on Surfaces*.⁹³ Since the angle trisection is incomplete without this lemma, and since it conforms with the special formulation of the problem appearing in it, we must suppose that both bore some relation to this lost Euclidean analytic work.

We face the problem, however, that nearly a century after Euclid, the parabolic case of this locus is constructed by Diocles without indication of awareness of a prior treatment.⁹⁴ Although Diocles might merely have been ignorant of Euclid's treatment, it seems better to suppose that he did indeed play a part in the discovery of this locus. But what part? Certainly, if the treatment of the parabolic case given by Pappus were known earlier, the solution of Diocles, which is essentially identical to it, could hardly have been original. Is it possible, then,

that an alternative method was modified by Diocles, thence to serve as a basis for the version given by Pappus?

A consideration of the Euclidean work will assist us. What was a "surface locus"? It is natural to suppose that it had to do with the tracing of curves on given surfaces, as via the intersection of two surfaces, and this view is in accord with remarks by Pappus and Proclus.⁹⁵ If so, we may perceive in the cube duplication of Archytas and the hippopedes of Eudoxus instances of such curves, and the former enters as a locus directed toward the solution of a problem. Other instances arise in Pappus: one may trace spirals on the surface of cylinders, cones, and spheres via projections of the plane spiral. While one usually assigns the definition of the latter to Archimedes, it is clear from Pappus that this curve was known to Conon somewhat earlier, so that its introduction and first study might well have occurred in Euclid's time.⁹⁶ To illustrate surface locus, Pappus describes the curve formed by intersecting a plectoid surface (i.e., a spiralling ramp) by an oblique plane; the orthogonal projection of the plane curve so formed is the quadratrix.⁹⁷ Pappus' method is analytic: assuming E on the quadratrix AK , the ratio $EZ : \text{arc } DG (= AB : \text{arc } AG)$ is given; for θ on the spiral, the ratio $\theta D : \text{arc } DG (= AM : \text{arc } AG)$ is given (see Fig. 17). Setting $EI = \theta D$, the ratio of $EI : EZ$ is thus given (Pappus does not state its actual value, $AM : AB$), so that I is given as on the intersection of the surface with a given plane through ZI . One may note that this alternative form, like the "mechanical" mode it would supplant, still requires appeal to continuity for determining its terminal position, the key to its use for the circle quadrature.⁹⁸ Doubtless, the projective form arose in connection with other applications of the curve, such as its use for the division of angles. Euclid himself is likely to have known of such uses through the efforts of predecessors like Dinostratus.

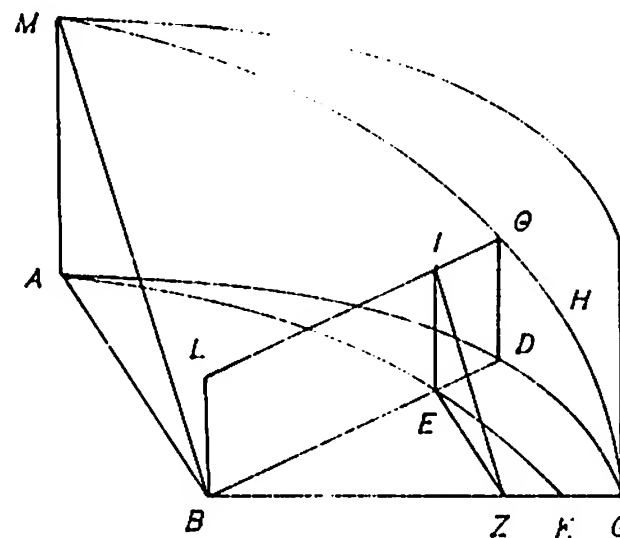


Figure 17

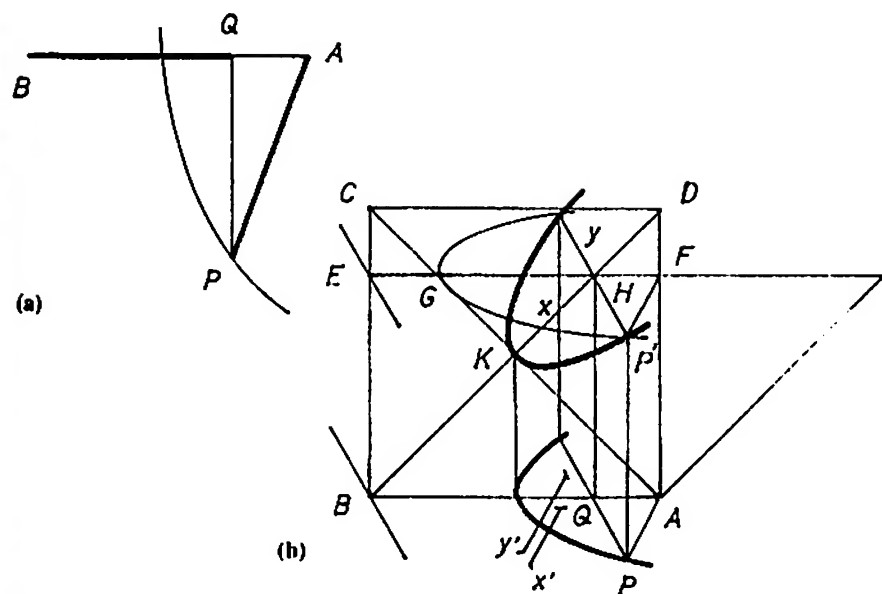
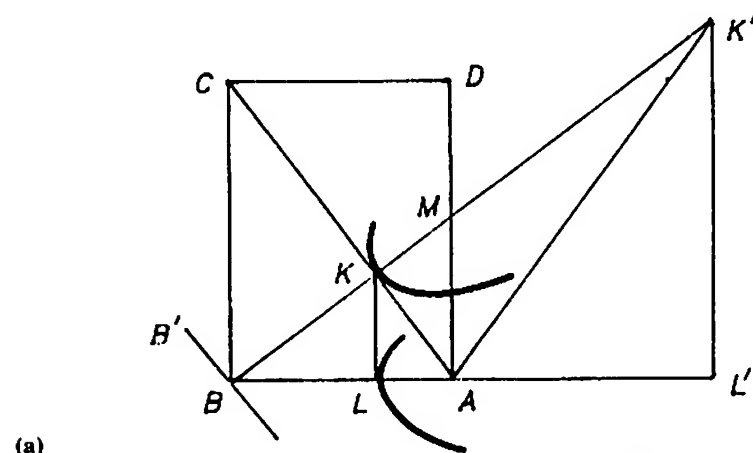


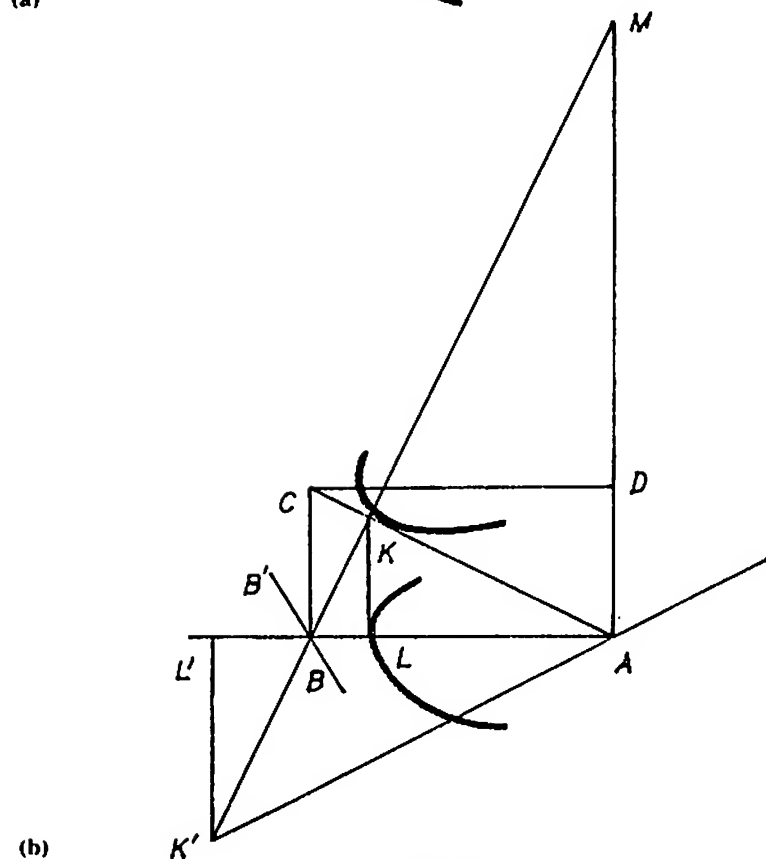
Figure 18

Using a method directly suggested by the solid projections presented by Pappus for the quadratrix, we can construct a solution to the focus-directrix problem in the following manner: first consider the case where the given ratio is 1:1, the given points A and B; we wish to find the locus of P such that $AP = BQ$, for Q, the foot of the perpendicular to AB from P (see Fig. 18). Let the square ABCD lie in a plane perpendicular to the given plane, and the line EF in a plane parallel to the given. Then the locus of points at distance AF from F is a circle which meets AC, the diagonal of the square at G; while the locus of points at distance EB (= AF) from the line through E perpendicular to BC in the elevated plane is a line parallel to that line meeting the diagonal BD at H. If we now consider all positions of the horizontal EF, the first locus will be a cone with vertex at A, axis AD, one generator coinciding with AC, and the vertex angle a right angle; while the second locus will be a plane passing through the line BD and containing the line perpendicular to it in each elevated plane. Since the lines BD, AC meet at right angles at K, the intersection of the surfaces will be a parabola, "the perpendicular section of the right-angled cone," and if we express it as $Y^2 = N \cdot X$ via coordinates in the oblique plane, then the line of application N will equal $2DK$, as we saw above. Now, if we project this parabola orthogonally onto the original plane, lines AP and BQ will be equal, for the associated lines FP' and EQ' (that is, EH), respectively equal to them are equal to each other in the elevated plane. Thus, P lies on the required locus.⁹⁹ Moreover, the projection leaves Y unaltered, but shrinks X by the factor $1 : \sqrt{2}$, so that the curve will be expressed by $Y'^2 = N' \cdot X'$, for $N' = 2\sqrt{2} AK$ or $2AB$. Thus, the required locus is a parabola of parameter twice the given line.

The cases of the hyperbola and the ellipse are analogous, each utilizing a rectangle ABCD in place of the square used for the parabola. Let $AL : LB$ be in the given ratio, and draw the right triangle ABK such that KL is the altitude to the hypotenuse AB (see Fig. 19). Then if we complete the rectangle ABCD



(a)



(b)

Figure 19

such that AKC is its diagonal, $(AB : BC)^2 = AL : LB$, the given ratio. We designate the line BB' perpendicular to AB in the base plane perpendicular to that of $ABCD$. If we now introduce the cone of axis AD and generator AC , it will intersect the plane of the lines BK, BB' in a conic section, each of whose points is such that its distances from the line AD and the plane of BC, BB' are in the given ratio $AL : LB$. Setting K' as the point where BK extended again intercepts the cone (in the case of the hyperbola, its opposed branch) and by M the point where BK meets the axis AD , we have the diameter KK' and the parameter $2KM$ for this conic, such that the coordinates X, Y for each of its points satisfy the relation $Y^2 : X(KK' \pm X) = 2KM : KK'$. If we project this conic onto the base plane, the projected ordinate $Y' = Y$, while the projected abscissa

$$X' = \left(\frac{LB}{BK}\right) X.$$

The points on this projected curve satisfy the locus condition that their distances from the point A and the line BB' are in the given ratio $AL : LB$. Since

$$Y'^2 : X' (LL' \pm X') \left(\frac{KB}{BL}\right)^2 = 2KM : KK' = 2AL : LL',$$

and $KB^2 : BL^2 = AB : BL$, it follows that

$$Y'^2 : X' (LL' \pm X') = 2AB \left(\frac{AL}{LB}\right) : LL'.$$

Thus, the curve solving the locus is the conic of diameter LL' and parameter

$$2AB \left(\frac{AL}{LB}\right).$$

The view that Euclid's *Surface Loci* took up the solution of problems via the intersections of solids, in a manner like that given here, raises interesting possibilities concerning the nature of the *Porisms*. Consider, for instance, that in the above construction each of the lines BL', BK' is divided harmonically; that is, $BL' : RL = AL' : AL$ and $BK' : BK = MK' : MK$. Several of Pappus' lemmas to the *Porisms* intend to show that this harmonic relation is preserved through projection (Lemma 19; cf. 5, 6). Thus, one can see how such lemmas on solid projections might find their use within the study of geometric problems. The harmonic relation is especially important for the investigation of the tangents to the conics. Indeed, by virtue of this division in the case of lines BL', BK' above, it follows that the tangents drawn to the conics at the points corresponding to the abscissas AL, KM will intercept their respective diameters at the point B . This property of the tangent lines is proved by Apollonius in *Conics* I, 34 (cf. 33 for the parabola), but one can conceive an alternative derivation via solid methods. First consider the case of a circle to which are drawn the two tangents KA, KA' [see Fig. 20(a)]; if any secant line from K meets the circle in P, Q and the chord AA' in N , then $KP : KQ = NP : NQ$ (or equivalently, KN is the

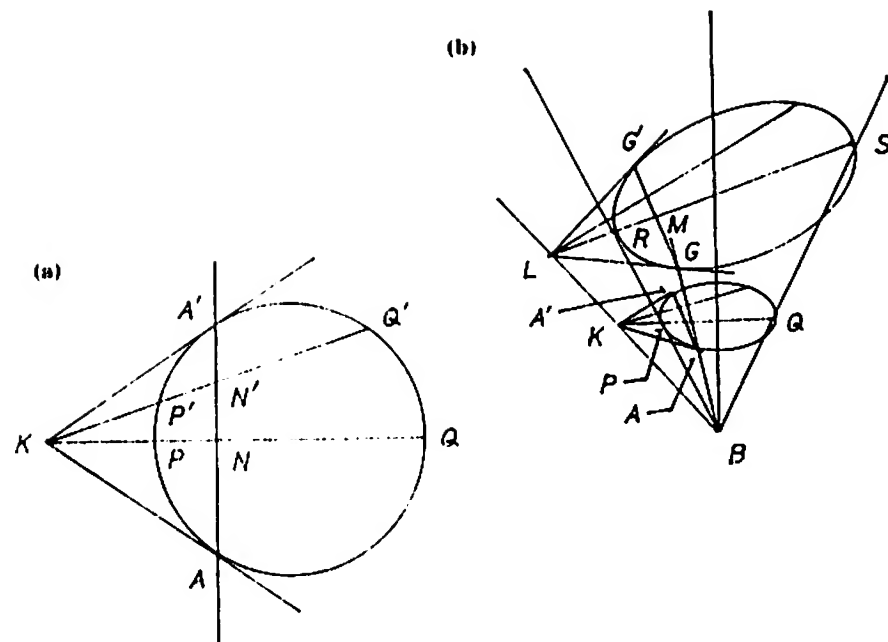


Figure 20

harmonic mean of PK, KQ).¹⁰⁰ Pappus includes a proof of this among the lemmas to the *Porisms* (Lemma 28).¹⁰¹ Let us next consider the case where PQ is a diameter of the circle, and let us view the circle as the section of a right cone of vertex B [see Fig. 20(b)]. Then if the figure is projected onto an oblique plane from the center B to yield a conic of diameter RS , the projected lines $GL, G'L$, corresponding to KA, KA' , respectively, will be tangents to this curve (for incidence relations are preserved), and the line $LRMS$ will be divided harmonically in correspondence with $KPNQ$.¹⁰² Hence, the harmonic property of the tangents to conics follows as a projective consequence of the property for the circle. At the same time, the harmonic division of any secant to a conic likewise follows from the case of the circle.

With reference to the same circle, we may ask for the division of secants corresponding to the geometric and arithmetic means [see Fig. 21(a)]. Since AK is the geometric mean of the segments PK, KQ of any secant, it follows that the locus of points J on the secants such that KJ is the geometric mean of PK, KQ will be the circle of center K and radius KA . The points I which mark off KI as arithmetic mean of PK, KQ will be the midpoints of the chords PQ , and hence lie on the circle of diameter OK (for O the center of the given circle), since the angle OKI will always be a right angle. If now, as before, the circles are conceived as sections of a solid, they can be projected onto oblique planes and so yield determinations of the conics whose points divide the secants to given conics according to the specified conditions of harmonic, geometric, and arith-

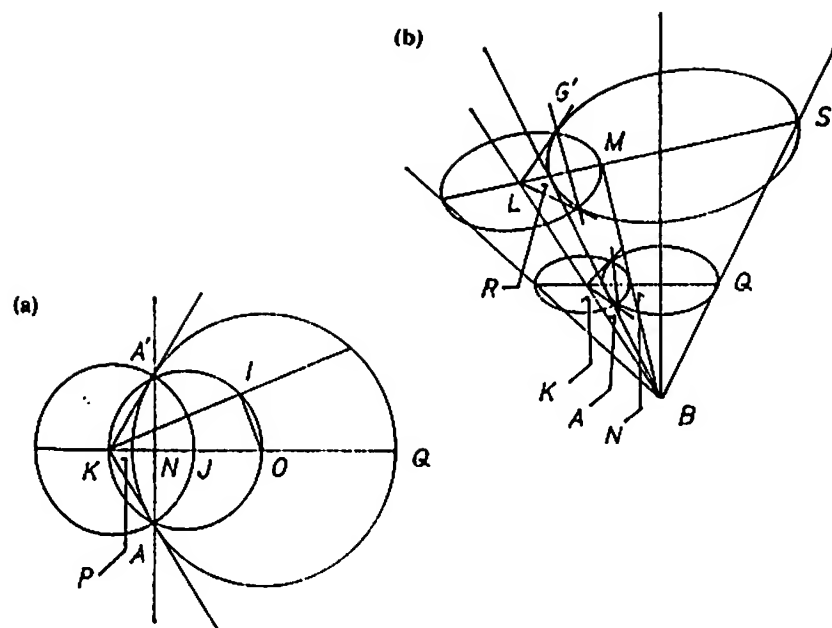


Figure 21

metic division [see Fig. 26].¹⁰³ Later we shall encounter a speculation by Zeuthen assigning to Eratosthenes an inquiry into locus problems of this type. The present remarks show how naturally the investigation of solid projections could develop in this direction.

Of these three locus propositions with reference to the circle, Pappus gives only that one relating to the harmonic property, but not those relating to the geometric and arithmetic means. Nevertheless, he does provide, in two other lemmas, certain other properties of a comparable kind (see Fig. 22):

Lemma 33: If points N, K lying in the line of the diameter of a circle are such that NK is the geometric mean of PK, KQ, and K' is any point on the line

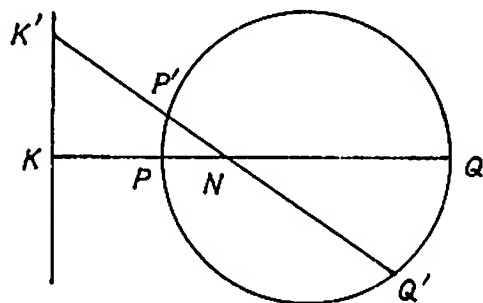


Figure 22

perpendicular to QK at K, then any secant K'P'Q' passing through N is divided such that NK' is the geometric mean of P'K', K'Q'.

Lemma 35: If in the same figure it is assumed that NK is the harmonic mean of PK, KQ, then also NK' will be the harmonic mean of P'K', K'Q'.

One notes that just as the earlier lemma (28) on the harmonic mean produced the construction of the polar line with respect to the pole K lying outside the circle, so in Lemma 35 one has the construction when the pole N lies inside the circle. In both cases one can obtain via solid projection in the cone an alternative proof of Apollonius' corresponding propositions for conics (III, 37-40).

Again consider the section of a right cone by a plane perpendicular to a generator (see Fig. 23). We have seen that if the diameter of the section PQ meets the axis of the cone in N and the base plane in K, then the line KPNQ is divided harmonically; and that if AN is an ordinate, the tangent at A passes through K. Now let any other plane containing line KK' perpendicular to KB in the base plane intersect the cone in the section of diameter P'Q'. Using B as the center of projection, we find that KP'N'Q' is also divided harmonically. Thus, if A' is the projection of A, the line KA' is tangent to the projected conic. The converse suggests a problem: to find the locus of points of tangency A

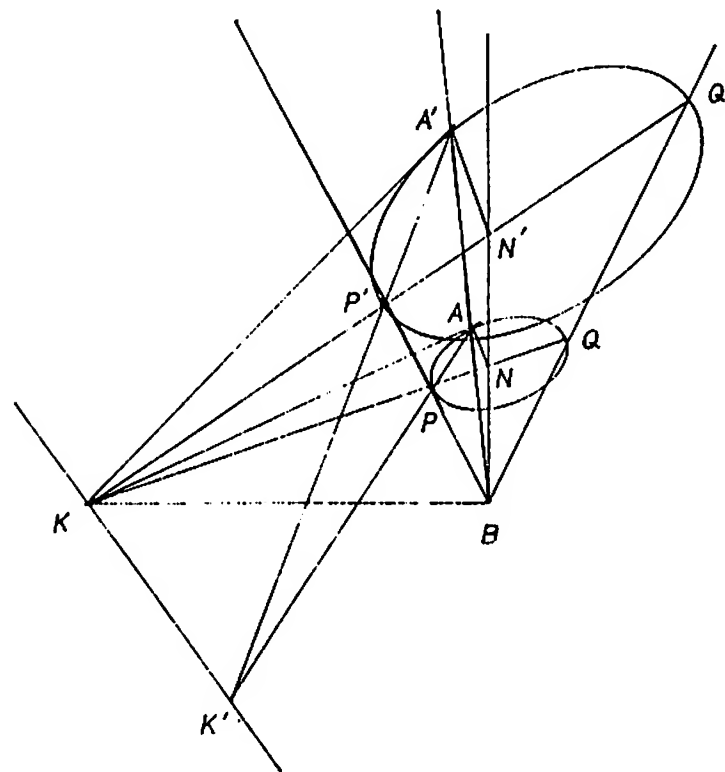


Figure 23

passing through a given line KK' . One approach might consider the projection of one section PQ onto another $P'Q'$ via the center B . As we have seen, since $KPNQ$ is divided harmonically, so also is $KP'N'Q'$. Since KA, KA' are tangents, the lines $AN, A'N'$ are each parallel to KK' and hence to each other. Since, furthermore, $AN, A'N'$ are radii of the circular sections of the cone made by planes parallel to the base through points N, N' , respectively, it follows that $AN : A'N' = BN : BN'$. Hence, the triangles $BAN, BA'N'$ are similar, so that BAA' is straight. Thus, the sought locus of points of tangency A is the generator of the cone BAA' extended. One may note the corollary, that if the lines $AP, A'P'$ are drawn and extended, they will meet at a point on KK' ; for the triangles $APN, A'P'N'$ are in projective correspondence in accordance with the configuration of Desargues.¹⁰⁵ Thus, for all curves in this family of sections, the line AP will pass through the same point K'' on line KK' .

These observations on the possible nature of Euclid's investigation of surface loci suggest a route toward the resolution of several puzzles surrounding his *Porisms*. In view of Pappus' lemmas to this work, we would gather that Euclid either assumed them without proof or else used a different method of proof from that given by Pappus. The context of stereometrical problems offers an appropriate set of techniques for such alternative proofs in contrast with the metrical procedures of plane geometry adopted by Pappus. Furthermore, the lemmas reveal the affinity of the subject matter of the *Porisms* with that commonly encountered now in the field of projective geometry. Yet it is clear that the ancients never introduced the conceptions and techniques of projectivity fundamental to the modern field, and so could never have undertaken the systematic exploration of the relations organized within that field. What, then, were the initiating interests which led them into these materials? We have seen how the study of problems relating to conics, as for instance the drawing of tangents, could give rise to configurations significant within the projective field. Among these are the two-triangle figure of Desargues, the complete quadrilateral, the harmonic division of a line and the preservation of this division under projection. For these have natural interpretations in the context of problems involving the formation of conic curves as different planar sections of the same cone.

If through his *Porisms* and *Surface Loci*, Euclid did indeed attempt to inaugurate a stereometrical study of the conics, his approach was soon modified drastically. This three-dimensional aspect is of course still in evidence in Archimedes' studies of the conics of revolution; but they have all but disappeared from Apollonius' theory of conics. For here the formation of these curves as conic sections serves only to fix their conception and to permit the derivation of the planimetric second-order relations of their ordinates. Thereafter, all properties and applications of the curves follow from these relations via the manipulation of proportions of lines in the plane.¹⁰⁶ For instance, the problem of the locus relative to a point and a line, which we solved above through the sections of a cone, is solved planimetrically in an alternative method preserved by Pappus; here the condition of the locus is transformed so that either $Y^2 = 2AB \cdot X$ or

AB and the given ratio $AB : BB'$. It is not difficult to perceive how a stereometrical treatment comparable to that reconstructed above might have served as a precedent for the extant planimetric treatment. If Euclid had solved the locus problem in the former manner, perhaps even omitting the parametric expressions for the solving conics, there would have been ample cause for Diocles and for the author of the lemmas to the *Surface Loci* to produce their alternative solutions. In this way, we can reconcile Euclid's awareness of the focus-directrix property of the conics with their re-examination by these later writers.

EUCLID'S CONTRIBUTIONS TO THE STUDY OF PROBLEMS

None of the lost mathematical writings alluded to in ancient sources seems more enigmatic than Euclid's *Porisms*. Pappus' extensive lemmas and commentary on the work have permitted detailed and persuasive reconstructions, especially that by Chasles. But its underlying motive and the scope of the wider geometric field toward which it was compiled remain unclear. The ambitious view advocated by Zeuthen seizes upon the striking correspondences with the modern field of projective geometry to assign to Euclid and his followers the elaboration of a projective theory of the conics, culminating in the projective generation of conics with reference to five given points, a result equivalent to Pascal's "mystic hexagon." Unfortunately, the ancients never introduce the general conceptions essential for the rationale of projective geometry. I have attempted an intermediate position, suggesting that Euclid experimented with a stereometric approach to the study of conics. His own *Optics* provides a ready context for such an inquiry; for instance, his observation that obliquely viewed circles appear "pressed in" (Prop. 36) at once raises the question of the precise determination of the apparent curve, whose solution as an ellipse actually appears among Pappus' lemmas to this work.¹⁰⁷ My view would thus assign to Euclid a form of the study of conics not unlike that later adopted by Desargues and Pascal, in which propositions established for circles are extended to conics via solid projection.¹⁰⁸ This does not intend to deny that the projective *conceptions* were for the most part absent from the ancient work and that these were thus truly novel elements of the 17th-century efforts. Nor do the several reconstructions of conic theorems given above presume to exhibit the method actually adopted by Euclid and his followers, but rather serve as specimens of a field of results readily accessible to them upon the adoption of such a stereometrical approach. The fact that Desargues and Pascal, firmly grounded in the study of Apollonius and Pappus, moved in this same direction suggests that they might in part have been rediscovering forgotten aspects of the ancient work in the process of advancing their own. Our task of interpreting the ancient field will surely have a more appropriate instrument in these 17th-century efforts than in the more sophisticated studies by the 19th-century projective geometers.

It is thus clear that Euclid's study of the conics, his *Porisms* and *Surface*

Loci, together with his *Data* and *Elements*, had special utility for the analytic investigation of geometric problems. The study of locus problems appears to have attracted special interest at this time. We have seen examples in the two-point locus of the Aristotelian *Meteorologica*, the point-line locus (or the specification of a conic via focus and directrix, as in the lemmas to the *Surface Loci*), and the locus with reference to three and four lines; doubtless many more of comparable type were then examined, although our sources no longer preserve clear information on the wider scope of this activity.

As for Euclid himself, one surely does better to view him as an effective teacher and compiler than as a phenomenally gifted mathematical intellect in his own right. The *Elements*, for instance, drew on the discoveries by Theaetetus and Eudoxus and their followers, while its individual books appear to have already received a systematic organization in the decades before Euclid. The *Data* was novel in form only, its content largely duplicating that of the *Elements*, but cast specifically for application in analyses. In the area of the conics, Euclid had the work done by Aristaeus as a precedent. We must regret not knowing more about the *Surface Loci* and its precursors; but, presumably, work extending as far back as Archytas and Eudoxus, in their studies of the cube duplication and other solid constructions, could serve as background. As for the *Porisms*, however rich the potential field might have been relating to it, Euclid's treatment had its weaknesses. For instance, Pappus was able to formulate into single propositions what Euclid presented in as many as ten and could see how some of Euclid's results extend into more complicated configurations in a way not noted by Euclid.¹⁰⁰ The proliferation of special cases which seems to have marked Euclid's approach must surely have obscured the general patterns in this material. But Apollonius charges Euclid not only for having taken up merely "the chance portion" of the locus of three and four lines, but also for having done so "unsuccessfully." These criticisms are implicitly admitted by Euclid's apologists, in a passage cited by Pappus, in that they insist he made no presumption of having exhausted these fields, but that his effort proved to be an indispensable basis for further research, including that of Apollonius himself.¹⁰¹

We should keep in mind, then, that in Euclid's time the theory and application of the conics were still relatively new, constituting a field undergoing its first consolidation. His contributions were seminal for the development of problem solving in the 3rd century. But it is entirely unfair of us to demand that his own efforts here matched the refinement of the *Elements* or of the Apollonian *Conics*.

NOTES TO CHAPTER 4

¹ The biographical data on Euclid are meager. His activity at Alexandria is attested in passages by Proclus and Pappus; but the former is anecdotal ("no royal road to geometry" addressed to King Ptolemy), while the latter refers to Euclid's *students* at Alexandria (cf. T. L. Heath, *Euclid's Elements*, I, Ch. I for references and discussion). On the other hand, the nature of Euclid's work well suits the environment of Alexandria: his several textbooks on geometry and mathematical science would find a welcome place in the research and teaching at the Museum and use to the fullest advantage the textual

resources of the Library. One usually sets his activity near the time of the founding of the Museum, thus centering on the first two decades of the 3rd century B.C. Note, however, that Archimedes' earliest recognizable citation of Euclid's *Elements* is in *Spiral Lines*, well after the middle of the century, so that a later dating may be indicated for Euclid (see my "Archimedes and the Elements" and "Archimedes and the pre-Euclidean Proportion Theory"). Recent authors wish to deemphasize the role of Alexandria as a center for mathematical and astronomical study; cf. G. J. Toomer, *Diocles*, p. 2 and O. Neugebauer, *History of Ancient Mathematical Astronomy*, pp. 571 f.

² Heath, *History of Greek Mathematics*, I, pp. 357 f. See also his survey of major editions of the *Elements* from antiquity to the early 20th century in *Euclid's Elements*, I, Ch. VIII. In addition to Heath's ample notes on the *Elements*, one may consult the studies by E. Neuenschwander, "Die ersten vier Bücher der Elemente Euklids," *Archive for History of Exact Sciences*, 1973, 9, pp. 325-380 and "Die stereometrischen Bücher der Elemente Euklids," *ibid.*, 1974-75, pp. 91-125; and I. Mueller, *Philosophy of Mathematics and Deductive Structure in Euclid's Elements*, Cambridge, Mass., 1981. The background to the Euclidean theory of incommensurables is examined in my *Evolution of the Euclidean Elements*, Dordrecht, 1975, while a survey of Book X of the *Elements* is provided in my "La Croix des Mathématiciens."

³ On this technique, see my *Evolution of the Euclidean Elements*, Ch. VI, Pt. IV. The standard account derives from H. G. Zeuthen; see also Heath, *History*, I, pp. 150-154 and Apollonius of Perga, pp. cii-xi.

⁴ An account of the relation of the *Elements* to prior efforts appears in my *Evolution*, Ch. IX, with references there to other discussions.

⁵ Proclus, *In Euclidem*, ed. Friedlein, p. 426. See Heath's account of the theorem in *Euclid*, I, pp. 350-354. On its Babylonian appearances, in particular in the "Pythagorean triplets" on the cuneiform tablet Plimpton 322, see O. Neugebauer, *Exact Sciences in Antiquity*, pp. 35-40 and B. L. van der Waerden, *Science Awakening*, pp. 76-80.

⁶ See my "Archimedes and the Pre-Euclidean Proportion Theory" and *Evolution*, Ch. VIII, Pt. II and Appendix B.

⁷ On irrationals see my *Evolution*, Ch. VIII, Pt. IV. On the pre-Euclidean study of the postulate, see I. Toth, "Das Parallelenproblem im Corpus Aristotelicum," *Archive for History of Exact Sciences*, 3, 1967, pp. 249-422.

⁸ Aristotle, *Opera* (Bekker), 376 a 4-6. I have altered the punctuation slightly. Note that the writer tacitly assumes that $MK = KH$ (being radii of the same circle) in order to infer that the ratio $HM : MK$ is given.

⁹ Apollonius' proof, to be discussed below, is given in full by Eutocius in his commentary on the *Conics*; see Apollonius, *Opera*, ed. Heiberg, II, pp. 180-184. That this result was proved by Apollonius in the second book of the *Plane Loci* is asserted by Pappus, *Collection* (Book VII), ed. Hultsch, II, p. 666. Two detailed accounts, establishing the identity of the two methods and giving full technical and textual comparisons, are presented by Heath, the first in his *Euclid*, II, pp. 197-200 (commenting on VI, 3), the second in his *Mathematics in Aristotle* (1949), pp. 181-190. The latter account was motivated by Heath's regret that writers on the Aristotelian meteorology had overlooked his earlier analysis and so erred in their discussion of this passage. It is to be all the more regretted that the Loeb translator of the *Meteorologica* repeated the oversight a few years later (1952). F. Solmsen's hesitation to accept the accuracy of Heath's position on this passage is likewise regrettable (*Aristotle's System of the Physical World*, 1960, p. 420n).

" I follow the construction in the *Meteorologica*, but have partially adapted the latter part of the proof where it seems muddled and have avoided its use of an indirect mode of reasoning.

" It is possible that this slip, together with the adoption of the indirect mode of proof, might indicate the author's having confused this construction with the proof of its converse, the completeness of the solution. For the proof that *all* points satisfying the condition of the locus lie on this circle is indeed well suited to an indirect demonstration, as one sees from the treatment given by Apollonius. Note further that the point *n* here cannot be the same as *N* in Fig. 4, even though the author seems to view it so. Apparently, he has confused two senses of the phrase "the arc through *M*."

" Heath notes this detour (*Elements*, II, p. 200) without registering surprise at the monumental lapse implied in the Apollonian treatment.

" Heath summarizes Simson's method in both of his accounts.

" This relation, not in the *Elements*, is assumed by Pappus in the *Collection* (Book VII), II, p. 730.

" Oddly, Heath persists in following Simson's approach in attempting his own reconstruction of the analysis underlying the *Meteorologica* passage. Furthermore, he seems to assume that one knows beforehand that the locus will be a circle. But surely, this fact too must be revealed by the analysis, not just its center and radius. These lapses detract from the effectiveness of his account for displaying the line of thought actually underlying the Aristotelian/Apollonian approach.

" It is remarkable that Pappus adopts much the same method in the solution of a related problem. In Lemma 29 to Euclid's *Porisms* (*Collection* (VII), II, pp. 904-906), he seeks to inscribe a broken line AGB within a given segment of a circle, such that AG : GB is a given ratio (Fig. 24). Thus, he draws the tangent line GD, whose length is the mean proportional between BD, DA, whence $AD : DB = AG^2 : GB^2$. Since AG : GB is given, so also is AD : DB; since D is thus given, so also is G. Here, the tangent line corresponds to PM in the Aristotelian/Apollonian locus problem. In effect, Pappus' method finds the solution via the intersection of two loci: that answering to the given ratio and that answering to the given angle of separation of AG, GB (i.e., the given circular arc). As it happens, al-Birūnī solves this same problem in his *Book of Chords*

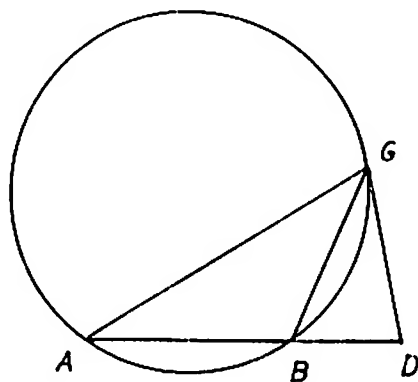


Figure 24

(cf. H. Suter, "Buch der Auffindung der Sehnen im Kreise von . . . el-Birūnī," *Bibliotheca Mathematica*, 1910-1911, II, p. 36); since the angle is given, he shows one need only construct a triangle with this angle at its vertex, such that the adjacent sides are in the given ratio, and then inscribe the similar triangle in the given segment. That Pappus follows such an elaborate alternative method here must indicate that he has lifted this solution from a context which treated it as a corollary to the two-point locus problem, the latter presented in the Apollonian manner. This may indicate that Euclid himself in the *Porisms* presented the locus problem in this same manner.

" The proof that this locus is a circle appears earlier in the account of the rainbow, *Meteor.* III, Ch. 3.

" Alexander of Aphrodisias (2nd century AD), *In Meteorologica*, ed. Hayduck (*Commentaria in Aristotelem Graeca*, III), pp. 162-172; Olympiodorus (6th century AD), *In Meteor.*, ed. Stüve (CAG, XII), pp. 243-259. Note that both commentators assume the simpler version of the uniqueness proof just given as Aristotle's intent (Olympiodorus, Lesson 38, pp. 245 ff). They should thus view the entire remaining portion of the geometric proof as superfluous. Yet each goes on to an elaborate line-by-line exegesis of that proof, without seeming to grasp what that proof is intended to establish (*ibid.*, Lesson 39, pp. 253-258). Note that some modern translators and commentators appear to be in the same situation (see note 9 above).

" This aspect of the passage was brought to my attention by H. Mendell.

" Pace Solmsen, who appears content with assigning the proof to Aristotle himself (*op. cit.*, p. 420).

" I touch on these passages in "Archimedes and the Pre-Euclidean Proportion Theory" and in an unpublished essay, "Zeno, Aristotle and Eudoxus." But they have not received serious attention among Aristotle scholars, including Ross (*Aristotle's Physics*) and Heath (*Mathematics in Aristotle*). Cf. my review of H. J. Waschkies, *Von Eudoxos zu Aristoteles*, in *Isis*, 1980, 71, p. 507.

" Cf. the pseudo-Euclidean *Catoptrica*, Prop. 30 (Euclid, *Opera*, ed. Heiberg, VII) and Diocles' *Burning Mirrors*, ed. Toomer, Prop. 1-3; here one investigates burning mirrors by which rays from the sun are reflected to a point or a line.

" Cf. Solmsen, *op. cit.*, p. 419.

" C. Boyer, *The Rainbow: From Myth to Mathematics*, 1959, pp. 39-54.

" That is, when the sun is rising or setting at the time the rainbow is seen. Generally, one must take ray MH parallel to the direction of the line from the observer at K to the sun.

" See H. D. P. Lee's introduction to the Loeb edition, pp. xxiii-xxv. Note that the authenticity of Book IV, on the nature of chemical processes, has long been doubted, although Lee defends it (*ibid.*, pp. xiii-xxi).

" Euclid also standardizes "right line" (*eutheia*) in preference to "line" (*grammē*) for rectilinear segments, and is followed in this by later authors.

" On Hippocrates, see Chap. 2 above. Besides the *Prior Analytics*, one may consult Aristotle's *Physics* (IV, ch. 8; VI, ch. 2, 4-7, 9) and *de Caelo* (I, 12; II, 4; III, 2) for this usage. Among later authors, only Philo of Byzantium adopts the same, to my knowledge. A mechanical writer late in the 3rd century AD, his account of the cube-duplication bears several indications of his distance from the field of geometric research at that time; the passage in Marsden (*Greek Artillery*, p. 110) may be compared with

accounts by Hero, a mechanical writer, like Philo, but much closer to the standard forms of geometrical writing (cf. Pappus, *Collection*, I, pp. 62–64).

²⁹ See note 11 above.

³⁰ Although we of course cannot know for certain the identity of the geometer who discovered this solution of the locus problem, it may well be pertinent to this question that Proclus says of Hermotimus of Colophon, a bit before the time of Euclid, that he “composed some things on loci” (*In Eucl.*, p. 67).

³¹ For references to Pappus and Proclus on the meaning of “porism,” see Heath, *History of Greek Mathematics*, I, pp. 431 f, 434 f.

³² Pappus’ commentary on this corpus occupies the entirety of the 238 propositions of the seventh book of his *Collection*, and is our principal, often unique source as to its contents.

³³ The *Data* has been edited in Greek by Menge (*Euclid, Opera*, VI), and its medieval Latin translation has recently been edited with English translation by S. Ito.

³⁴ See, for instance, Heath’s discussion of Prop. 93 (*History*, I, pp. 424 f).

³⁵ For authorities espousing this view, see Ito, *op. cit.*, pp. 11 f. But most commentators appear to have had a sense of the relevance of the *Data* for analyses; cf. Heath, *History*, I, p. 422.

³⁶ Aristotle notes that propositions are not always simply convertible, and sees that as a cause why analytic methods are more useful in geometry (whose propositions tend regularly to be convertible) than in dialectic (*Post. An.* I, 12). For a logical and historical account of this method, see J. Hintikka and U. Remes, *The Method of Analysis*, 1974 and M. Mahoney, “Another Look at Greek Geometrical Analysis,” *Archive for History of Exact Sciences*, 1968/69, 5, pp. 319–348. See also Chapter 3.

³⁷ Proclus informs that Leon, a contemporary of Plato, investigated diorisms to determine the limits of the solvability of problems (*In Euclidem*, p. 66). This bears on the *Meno* passage discussed in Chapter 3.

³⁸ See, for instance, the striking similarity between *Data*, Prop. 58 and *Elements* VI, 26 on the application of areas. On the other hand, some parts of the *Data* adopt a mode at variance with the *Elements*. The section on parallels, for instance, conceives of a “displacement” (*metapiptein*) of given lines (*Data*, Prop. 28) in contrast with the strictly static view in the *Elements* (I, 30). It may thus be that some portions of the *Data* had already been compiled before Euclid and were incorporated by him into a larger treatment, with his own additions based on the *Elements*.

³⁹ Cf. the problems in *Elements* I, 1–3, 9–12, 22–23, 31, 44–46; all of IV; further instances in VI, XI, XIII. A comparable usage occurs in the Aristotelian tract *On Invisible Lines* (970 a 8): to construct a triangle whose three sides are “given” (cf. *Elements* I, 22).

⁴⁰ *De Caelo* I, 2, 283 a 6; for other instances, see Bonitz’ index in Aristotle, *Opera* (Berlin edition), V, s.v. *diōmi*.

⁴¹ *Collection* (Book VII), II, p. 672; cf. p. 676 and below, note 57. We return to Aristaeus in Chapter 7.

⁴² Cf. *Quadrature of the Parabola*, Prop. 1–3. A survey of such instances is given by Heath, *Apollonius*, Ch. III. An effort to compile the whole of the Archimedean theory of conics was made by J. L. Heiberg, “Die Kenntnisse des Archimedes über die Kegel-

schnitte,” *Zeitschrift für Mathematik und Physik* (hist.-litt. Abth.), 1880, pp. 41–67. But its comprehensiveness has recently been set into question (G. J. Toomer, *Diocles*, p. 4).

⁴³ See Apollonius, *Conics* I, preface and Heath’s discussion, *Apollonius*, Ch. 1.

⁴⁴ Note that some parts of the earlier theory might happen not to appear in Apollonius. Such seems to be the case for the theorems on the focus-directrix construction of the conics (see Sect. iv below); cf. also Diocles’ assumption of the constancy of the subnormal of the parabola, only found in Apollonius’ Book V in somewhat different form (cf. Toomer, *op. cit.*, pp. 17, 151).

⁴⁵ For instances of such inquiries, see Chapter 3.

⁴⁶ “On the Astronomical Origin of the Theory of Conic Sections,” *Proceedings of the American Philosophical Society*, 1948, 92, pp. 136–138.

⁴⁷ *Collection* (Book VII), II, pp. 588 ff; see Heath, *History*, II, pp. 397 ff.

⁴⁸ [Aristotle], *Problems* XV, 7, XVI, 6; discussed by Heath, *Mathematics in Aristotle*, pp. 264–267. Note that *thyreas* was still admitted as a name for the ellipse by Geminus (1st century A.D.), as cited by Proclus (*In Euclidem*, pp. 111, 126). In his work on spherical astronomy, the *Phaenomena*, Euclid argues the sphericity of the realm of the stars on the basis that it always appears to produce a circle when sectioned by a plane, while in the case of cylindrical and conical figures the result would be an ellipse, that is, “the section of an acute-angled cone” (Preface, *Opera*, VIII, pp. 4–6). Euclid is surely mistaken in alleging that the appearances of these three sections will be different; for the observer will view them from within the plane of their formation, so that the sections of the cylinder and the cone will appear precisely superimposed over the section of the sphere, giving all three the appearance of circles. It is surely significant that Ptolemy does not use this argument in his account of the sphericity of the cosmos (*Syntaxis* I, 3).

⁴⁹ Cf. Proclus, *In Euclidem*, p. 356: one deduces the “symptom” (property) of the curve as the basis of its study. This characterizes Archimedes’ treatment of spirals (*Spiral Lines*, Prop. 2, 12; Pappus, *Collection* (IV), I, p. 234), as also Nicomedes’ of the conchoids (Pappus, *ibid.*, p. 244) and that of the quadratrix (*ibid.*, p. 252). This distinction between generating mode and property has been exploited by Zeuthen in his argument that the ancients intended problems of construction to serve as existence proofs (“Die geometrische Construction als ‘Existenzbeweis’ in der antiken Geometrie,” *Mathematische Annalen*, 1896, 47, pp. 222–228); I raise reservations concerning this view in Chapter 8.

⁵⁰ Such a construction was used by J. Werner in his *Elements of Conics* (1522), Prop. XI; see M. Clagett, *Archimedes in the Middle Ages*, IV, Pt. I, pp. 254, 277, 298.

⁵¹ Diocles and Eutocius describe such pointwise procedures; see Ch. 3, notes 36–37.

⁵² Pappus, *Collection* (VII), II, pp. 672–674; Eutocius, *In Apollonium*, ed. Heiberg, II, pp. 168–170. Note that even those writers like Euclid and Archimedes who subscribed to the older mode were still quite aware of the nature of oblique sections; cf. Heath, *Apollonius*, pp. xxxvi, xlv, and also note 48 above.

⁵³ See Toomer, *Diocles*, pp. 9, 14, Geminus, and after him Proclus, sometimes use the older terminology, perhaps a suggestion of their reliance on early sources; cf. *In Euclidem*, p. 111.

⁵⁴ For a similar intuition of the role of the solid construction within the early theory of conics, see van der Waerden, *Science Awakening*, p. 245 and H. G. Zeuthen, *Lehre von den Kegelschnitten*, pp. 467–469. The view is also sketched in Zeuthen’s “Con-

struction als Existenzbeweis" (see note 49 above). One should note that all these allegedly aim to provide a rationale for the *Menaechmean* conception of the conics.

⁵¹ This interpretation of the expression conforms with the view of Zeuthen and Heath; cf. Heath, *Archimedes*, pp. clxvii-iii; and Toomer, *op. cit.*, p. 13.

⁵² Cf. Toomer, *op. cit.*, pp. 5-7 and Heath, *Apollonius*, Ch. III and pp. lxxix ff.

⁵³ Pappus, *Collection* (VII), II, p. 674; cf. Heath, *Apollonius*, pp. lxxxiii-iv.

⁵⁴ One may consult the derivations given by Toomer (*op. cit.*, pp. 10-13) and E. J. Dijksterhuis (*Archimedes*, pp. 58, 59) for a proof that line C, as shown in Figs. 6(b), 6(c), is the line of application (the "latus rectum") in the parametric expression of the conics.

⁵⁵ Apollonius, *Conics* I, Preface; cf. Pappus, *Collection* (VII), II, pp. 672 ff and Heath, *Apollonius*, pp. xxxi-ii. Eutocius misconstrues this problem as referring to the finding of the two mean proportionals (Apollonius, *Opera*, II, p. 186).

⁵⁶ For a discussion and reconstruction of the Apollonian investigation of this locus, see Heath, *Apollonius*, Ch. V and Zeuthen, *Kegelschnitte*, Ch. 7, 8.

⁵⁷ Pappus, *Collection* (VII), II, pp. 648 ff; discussed by Heath, *History*, I, pp. 431 ff.

⁵⁸ This follows Simson's account, as reported by I. Thomas, *History of Greek Mathematics*, I, pp. 482-484. See also van der Waerden, *op. cit.*, pp. 287-290.

⁵⁹ For a compact survey of the history of projective geometry, see L. Cremona, *Elements of Projective Geometry*, 1893, pp. v-xii. The classic treatment is by Chasles, *Aperçu historique des méthodes en géométrie*, 1837 (1875; 1889).

⁶⁰ Van der Waerden reconstructs a proof, *op. cit.*, pp. 287 f.

⁶¹ Pappus, *Collection* (VII), II, p. 884 (Lemma 12; Lemma 13 treats the nonparallel case). For the entire set of lemmas, see *ibid.*, pp. 866-918. Brief accounts appear in Heath, *History*, II, pp. 419-424 and van der Waerden, *Science Awakening*, pp. 287-290.

⁶² *Les trois livres des Porismes d'Euclide*, 1860. The plan of this reconstruction is adumbrated in his *Aperçu historique*, pp. 274-284. Heath's assessment of his effort, in light of those of Simson (1776), Zeuthen (1886), and others, appears in *History*, I, pp. 431-438.

⁶³ Cf. Heath, *History*, I, p. 437. Heath's principal reservation against accepting Chasles' reconstruction has hardly any standing at all: it is that Chasles makes out the propositions in Euclid's *Porisms* to be on a par with Pappus' lemmas, while one should expect that the lemmas were pitched at a far lower level than the theorems in the associated treatise. But if such is often the case with Pappus' lemmas, it seems not always to have been so. For instance, the *neusis* problem from Heraclitus (Prop. 72) is surely comparable to the *neusis* constructions in Apollonius' work; a hyperbola construction among the lemmas to the *Conics* (Prop. 204) is the effective equivalent of a construction now to be found in *Conics* II, 4 (see Chapter 7, note 33); the focus-directrix constructions for the conics (Props. 236-238) must match or surpass their correlates in the Euclidean *Surface Loci* (see below). In several instances, Pappus provides problems with full analysis and synthesis; Props. 85, 87, 105, 107-109, 117; and he sometimes offers not only one, but sometimes two alternative proofs of lemmas (e.g., Props. 35, 36, 39). This must surely indicate that the lemma had a certain interest in its own right. Finally, one may consider

the construction of the ellipse conforming to the obliquely viewed circle (*Collection* VI, Prop. 53); this is in fact far beyond the scope of the technical methods employed in the associated treatise, Euclid's *Optics*. We thus have no reason to be uncomfortable with the view that in the instance of an early work like Euclid's *Porisms*, Pappus might provide in his lemmas a modification, or even an improvement, on materials in the original.

⁶⁴ *Kegelschnitte*, Ch. 8, especially pp. 173-184; cf. Heath, *History*, I, pp. 437 f. Zeuthen shows how a certain porism enunciated by Pappus can lead to the specification of a conic projectively, via its property as a four-line locus (*op. cit.*, pp. 169 f; we discuss this locus below). In this way, one comes to the point of a verification of Pascal's theorem on the hexagon inscribed in a conic; for that can be seen as a consequence of the generation of a conic as the locus of the vertex of a variable triangle whose other two vertices move along given lines, while its sides turn about given points. In noting this connection with the porism, Zeuthen nevertheless acknowledges that the ancients were not likely to have recognized the Pascal property; for Pappus would surely have mentioned it were it known (*op. cit.*, pp. 495 f).

⁶⁵ *Collection* (VII), II, pp. 884-886.

⁶⁶ In the parallel case, the ratio becomes ZD : GZ in each instance, as Pappus establishes in his Lemma 11.

⁶⁷ H. S. M. Coxeter, *Projective Geometry*, New York, 1964, pp. 38 f; W. T. Fishback, *Projective and Euclidean Geometry*, 2nd ed., New York, 1969, pp. 67-69; D. J. Struik, *Analytic and Projective Geometry*, Reading, Mass., 1953, p. 66; B. E. Meserve, *Fundamental Concepts of Geometry*, Reading, Mass., 1955, pp. 61-63.

⁶⁸ For instance, the lemma on the ellipse in *Collection* VI, Prop. 53 (see note 67 above) assumes a result proved in Pappus' Lemma 28 to the *Porisms* (VII, Prop. 154). Heath discusses a remarkable coincidence between Lemma 31 (VII, Prop. 157) and Apollonius' III, 45 on the foci of a central conic (*Apollonius*, pp. xxxix f).

⁶⁹ *Collection* (VII), II, pp. 648-652.

⁷⁰ Cf. Heath, *Apollonius*, pp. 95 f, whose lettering I adopt in part.

⁷¹ Cf. *ibid.*, pp. xxv, xlviii.

⁷² Zeuthen does not propose a reconstruction of the three-line locus; Heath develops an unnecessarily complicated form based on *Conics* III, 54-56 (*ibid.*, pp. 122f). Thus, the present version is my own.

⁷³ That is, QV will bisect all chords parallel to PP'.

⁷⁴ This limitation within the pre-Apollonian researches is stressed by Zeuthen; cf. Heath, *op. cit.*, pp. lxxxiv ff, cxli.

⁷⁵ For details, see Heath, *op. cit.*, pp. 123-125. Although he derives his form of the three-line locus from III, 54-56 (see note 76 above), this is immaterial to the subsequent derivation of the four-line locus from the three-line locus.

⁷⁶ *Lehre von den Kegelschnitten*, Ch. 7, 8; summarized by Heath, *op. cit.*, Ch. v.

⁷⁷ Cf. Heath, *op. cit.*, pp. cxlv f.

⁷⁸ Cf. *ibid.*, pp. cxlvii-ix.

⁷⁹ That five points determine a conic is established by Apollonius in *Conics* IV, 25 using the harmonic division of secants drawn to conics; results of the latter type are likely to have been well known in the earlier theory (see below). To effect the locus in the

CHAPTER 5

Archimedes: The Perfect Eudoxean Geometer

Among the ancient scientists none has attained greater fame, either in his own time or in ours, than Archimedes of Syracuse. His immense achievements in pure geometry extended into the field of mechanics where his theoretical findings had consequences for the design of mechanisms. Siege engines constructed under his supervision held the Roman fleet at bay for three years during the ill-fated campaign which despite his efforts ended in the city's fall and his own murder in 212 B.C. In the popular view Archimedes thus became the prototype of the scientific wonder-worker, an image sustained by the ancient historians and elaborated beyond the limits of credibility by the legendizers.¹ But the Archimedes we meet in his own writings is, with but rare exceptions, exclusively the pure geometer. It is his place within the development of the ancient geometry which shall concern us here.

Archimedes was born near the time that Euclid died, around the second decade of the 3rd century B.C.² One might thus suppose that Euclid's work was a dominant influence in Archimedes' training as a geometer and that Archimedes would eventually enter and extend the field of study opened up by Euclid. But this turns out not to be the case. Points of detail relating to Archimedes' early studies and his technique of proportions reveal him to depend on a tradition alternative to Euclid's *Elements*, doubtless closer to the manner of the original Eudoxean treatments.³ In the area of mechanics which so absorbed his attention, Archimedes could find no precedent in Euclid's work.⁴ Yet, as we shall see, the area which Euclid did advance, the application of the method of analysis for the solution of geometric problems, is one which drew relatively little interest from

Archimedes. This split is striking, but not entirely surprising. Archimedes' father was an astronomer, hence competent in the Eudoxean geometric methods and quite capable of rearing his son on the older geometric treatises.⁵ Moreover, the separation of Syracuse from Alexandria might well sustain differences in research interests. It would thus appear that Archimedes was too close in time, but too far in distance to come under the spell of Euclid's particular brand of geometry.

Characteristic of the Eudoxean geometry is the application of an indirect method of limits for the measurement of curvilinear plane and solid figures. The centerpiece of this tradition is the issue of the quadrature of the circle. One perceives this problem to be an important factor in Archimedes' early geometric efforts, particularly the *Dimension of the Circle*, while it leads naturally into the surface and volume measurements carried through in his *Sphere and Cylinder I* and the studies of areas and tangents in his *Spiral Lines*.⁶ The same Eudoxean approaches find their application in the demonstration of the principles of equilibrium and center of gravity in *Plane Equilibria*. From this a powerful heuristic method emerges by which the content and centers of gravity of figures can be determined through the conceptual weighing of their constituent elements. Archimedes neatly summarizes the manner of its use in the *Method* for obtaining the results whose formal demonstrations he had already communicated in the *Quadrature of the Parabola*, *Conoids and Spheroids*, and the lost *Equilibria*.⁷ These formal proofs adhere meticulously to the style of limits Archimedes developed in refinement of the Eudoxean method. One thus perceives that the techniques and concerns of Eudoxean geometry run through the entire Archimedean corpus, as extant in Greek, and indeed define its subject matter at virtually every point.

By comparison, the investigation of geometric problems assumes a low profile at best in Archimedes' works. It is the concern of only one of the extant Greek writings, *Sphere and Cylinder II*, where six of the nine propositions are problems seeking the construction of segments of spheres according to specifications of size and proportion, and where the double method of analysis and synthesis is applied in each instance. Of course, Archimedes shows himself to be complete master of the technique. Moreover, works preserved now only through their medieval Arabic translations include interesting problem-solving efforts, for instance, an angle trisection implied in one of the propositions in the *Book of Lemmas* and a seven-section of the circle worked out in the *Inscription of the Regular Heptagon*. Both solutions depend on *neusis*, that is, conceptual manipulation of an idealized sliding ruler, and through this have a natural connection with certain constructions, also effected via *neuses*, introduced in the investigation of the tangents in *Spiral Lines*. Few in number perhaps, these constructions of problems set a firm precedent for the solution of problems via special curves, *neuses*, and other means by geometers in the following generation.

Thus, through a survey of Archimedes' contributions to the solution of problems we cannot hope to reveal the scope, or even capture the essence of his geometric achievement. For this must omit discussion of his advanced efforts in the measurement of conics and solids of revolution and of his inquiries into

geometric mechanics and hydrostatics.⁸ But in the context of our other discussions, the work of Archimedes ought to emerge as the product of a powerful geometric intellect, at once influential for the development of the field, yet oddly out of sympathy with its principal objectives.

CIRCLE QUADRATURE AND SPIRALS

No geometric results discovered by Archimedes were more widely known and used in antiquity than those derived from his study of the measurement of the circle. Later writers cited his *Dimension of the Circle* for the rule that the area of the circle is one-half its perimeter times its radius, analogous to the area rule for triangles; they cited it also for approximations: that the circle is $1\frac{1}{4}$ times the square of its diameter, and that its circumference is $3\frac{1}{2}$ times its diameter. While these three results are established in the three propositions, respectively, of the extant *Dimension of the Circle*, the ancients knew this work in a different, more substantial version; for, in particular, they could draw other results from it, such as the analogous area rules for sectors and perhaps also an inequality appropriate for estimating segments of circles.⁹ Nevertheless, the extant version clearly reveals the source of Archimedes' formal technique and his skill at adopting this for computational purposes.

Most accounts of Archimedes' works assign this writing to a time relatively late in his career. But this view is the consequence of a plain misunderstanding. When Heiberg proposed his chronological ordering of the Archimedean writings, he saw no clear indications of the placement of the *Dimension of the Circle*, save its obvious affinity in content with the measurements of spheres in *Sphere and Cylinder I*. He thus set it at the end of his list, followed by a question mark to indicate his uncertainty. In the frequently consulted survey of Greek mathematics by Heath, this list was reproduced, but without the question mark for this entry.¹⁰ Thus, the circle measurement was transformed into a late writing, reversing Heiberg's tentative association of it with the relatively early sphere measurement. Further consideration supports Heiberg's intuition, although he appears to have underestimated the early dating of the circle measurement.

What marks the *Dimension of the Circle* as especially early is its adherence to the elementary Eudoxean form of the method of limits. For in his other writings Archimedes introduces a refinement on this method in which the curvilinear figure to be measured is bounded above and below by figures of known measure; as one can arrange that the bounding figures differ from each other by less than an arbitrary preassigned quantity, an indirect argument establishes that the intermediate figure must equal a specified figure also known to lie between these bounds.¹¹ For instance, Archimedes proves in *Sphere and Cylinder I*, 5 that to a given circle there may be circumscribed and inscribed similar regular polygons whose areas have a ratio less than any preassigned ratio greater than unity. It follows that such bounding figures may be found to differ by less than a preassigned area. To establish the result in *Dimension of the Circle*, Prop. 1, one would need only to note that the area determined as one-half the circumference

times the radius of the circle is greater than any polygon inscribed in the circle, but less than any polygon circumscribed about it. For the area of a regular polygon is one-half its perimeter times its in-radius; while the perimeters of the circumscribed and inscribed polygons are respectively greater than and less than that of the circle, their in-radii respectively equal to and less than the radius of the circle. Thus, the area of the circle and the product of one-half its perimeter times its radius must differ by less than any preassigned quantity. Hence, as claimed, they are equal.¹²

Instead of the expected two-sided convergence method, Archimedes follows the Eudoxean one-sided method here, in which the lower bounding figures are considered entirely separate from the upper bounds. In each instance, doubling the number of sides of the polygon reduces at least by one-half the difference of its area from that of the circle.¹³ Through the same area rules for regular polygons stated above, it follows that the circle is neither less than nor greater than the specified product; hence it is equal to it. This procedure conforms precisely to the Eudoxean form of the circle measurement, as we know it from Euclid's *Elements* XII, 2, with but one important exception. The Eudoxean theorem has to do with the *ratios* of circles. In the indirect proof, the assumption of a lesser inequality can be reduced to a contradiction through consideration of inscribed polygons. But instead of introducing the circumscribed figures to eliminate the assumption of greater inequality, one may reduce this case to the prior case merely by inverting the ratios. This ploy is not available to Archimedes, since his theorem asserts an equality of areas rather than of ratios of areas. Introducing the circumscribed figures entails a further assumption on the magnitudes of arcs. Just as he had assumed that the perimeter of the inscribed polygon is less than that of the circle, so now he must assume that the perimeter of the circumscribed polygon is greater than that of the circle. The former assumption might be considered obvious by virtue of the conception of the straight line as the shortest distance between two given points. This notion was surely familiar in the pre-Euclidean period, but it is not articulated as an axiom by Euclid.¹⁴ To the contrary, an equivalent for the rectilinear case is established as a *theorem*: that in any triangle the combined lengths of any two sides must be greater than the third side (I, 20). Later philosophers criticized Euclid for belaboring the proof of what is obvious to any ass.¹⁵ But surely Euclid, and doubtless geometers before him, intuited the generality of this result. It is hard to suppose, moreover, that no attempt was made around the time of Eudoxus to establish the proportionality of circumferences and diameters in view of the analogues for circular area and spherical volume in XII, 2 and 18. As far as we know, the requisite axioms receive their first explicit formulation by Archimedes among the postulates prefacing *Sphere and Cylinder* I: that the shortest distance between two points is the straight line joining them; and that for two curves, convex in the same direction and joining the same two points, the one which contains the other has the greater length.¹⁶ In the *Dimension of the Circle*, however, one finds no specific indication that an axiom of this form underlies the steps that the perimeter of the inscribed polygon is less than that of the circle, while the perimeter of

the circumscribed polygon is greater. Indeed, in this work no justification for these steps is given at all. Presumably, then, Archimedes could accept them as obvious. In view of his strong dependence here on the earlier techniques of convergence, one need not doubt that in these other regards he could find certain precedents in attempts by earlier geometers to prove theorems on arc length.¹⁷ The formulation of these principles as explicit axioms in *Sphere and Cylinder* I would thus result from Archimedes' own later reflections on the formal requirements of such demonstrations.

The Archimedean area rule in conjunction with the Eudoxean theorem on the ratios of circles (*Elements* XII, 2) yields the result that the circumferences of circles have the same ratio as their diameters. Pappus provides such a proof within a series of lemmas on the ratios of circles and their sectors and segments,¹⁸ and we should expect to find it in the *Dimension of the Circle*, for Archimedes goes on in its third proposition to estimate the ratio of the circumference to the diameter, *not* the ratio of the areas of the circle and the square on its diameter. In the extant Prop. 2, the latter ratio is asserted to equal 11 : 14, but its "proof" actually depends on the result of Prop. 3. Clearly, the writing as we know it has been altered to its detriment through editorial revisions and scribal confusions.¹⁹ The lemmas on related properties of the circle which we find in Pappus and Hero, for instance, may thus derive from Archimedes' work before it had suffered these textual corruptions.

Despite such difficulties the essential skill and genius responsible for the computations in the third proposition are unmistakable. Archimedes seeks to establish not merely an approximation for the ratio of circumference to diameter, but upper and lower bounds on that ratio. As in the first proposition, the procedure is divided into two separate sections, one dealing with circumscribed figures (leading to the upper bound), the other with inscribed figures (for the lower bounds). This division masks the fact that the computational procedures are identical, each requiring the same property of the bisector of the angle of a triangle (cf. *Elements* VI, 3). Indeed, complete derivations for both procedures are provided, despite their equivalence. Such intermediate proofs often turn out to be the work of interpolators. But even if that is the case here, the editor has only adopted and supplemented the separation of cases fixed in the computation itself. The failure to exploit the potential for refining this procedure by means of the two-sided notion of convergence thus serves to affirm the view that the *Dimension of the Circle* was an early Archimedean effort.²⁰

The procedure adopted is as follows. First consider the upper bounds, for which we take the circumscribed regular hexagon as the initial bounding figure. Form the right triangle whose leg A is one-half of one of the sides of this hexagon, whose other leg B is the radius of the circle, and whose hypotenuse is C, so that the central angle formed by B and C is $\frac{1}{12}$ the full circle [see Fig. 1(a)]. If the corresponding element of the polygon having twice as many sides is denoted by A', B', and C', Archimedes' rule establishes that $A' : B' = A : B + C$, while $C'^2 = A'^2 + B'^2$. Starting from the hexagon, he can then proceed in succession to the polygons of 12, 24, and 48 sides, ending with that of 96

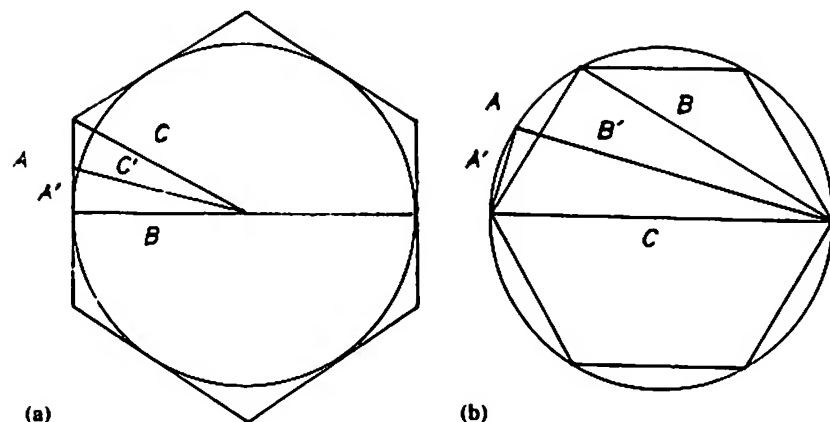


Figure 1

sides. In each case, rounding off due to the approximation of irrational square roots must give an upper bound on A (that is, a lower bound on B and C), thus ensuring that the final value of the ratio of $A : B$ will yield an upper bound for π , namely $96 \cdot 2A : 2B$. For the initial value of $A : B = 1 : \sqrt{3}$, he uses the upper bound $153 : 265$. The manner of its derivation is unexplained, and has given rise to many attempts at reconstruction by scholars tantalized by the fact that this is one of the convergent fractions in the continued-fraction development of the root.²¹ After four applications of the rule, Archimedes obtains $153 : 4673\frac{1}{2}$ as an upper bound on $A : B$, from which he produces $96 \cdot 153 : 4673\frac{1}{2} = 14688 : 4673\frac{1}{2}$ as an upper bound on π . The final value $3\frac{1}{7}$ follows from this through rounding off upward. Here again the effort to find good fractional approximations in the smallest possible terms appears to be assisted by procedures comparable to the development in continued fractions.²² The configuration for the lower bounds is altered, in that A is the side of the inscribed polygon, while C —rather than $2B$ —is the diameter of the circle [Fig. 1(b)]. Otherwise, the computation proceeds as before, where rounding off must always be downward so that the ratio $A : C$ will produce $96 A : C$ as a lower bound on π . The initial value for $A : B$ is $780 : 1351$ here (another convergent in the continued fraction for $1 : \sqrt{3}$), and after four applications of the rule, one obtains $66 : 2017\frac{1}{4}$ as lower bound for $A : C$.²³ Hence, $96 \cdot 66 : 2017\frac{1}{4} = 6336 : 2017\frac{1}{4}$ is a lower bound for π . This is simplified to $3\frac{1}{7}$ via another downward rounding off.²⁴

In this way Archimedes has established $3\frac{1}{7}$ as an upper bound and $3\frac{1}{7}$ as a lower bound on π . Although the computation requires that the error due to rounding off must be cumulative, the final values differ by less than one part in 2400 and one part in 4200, respectively, from the actual value. The Greek tradition is aware of no better estimate before the time of Archimedes, while the value $3\frac{1}{7}$ is employed without exception within the tradition of metrical geometry after him.²⁵ But closer estimates were derived through further appli-

cations of Archimedes' method. Hero reports that Archimedes himself produced a different set of bounds in a work *On Plinthis and Cylinders*. Eutocius cites a work called the *Easy Delivery* (*Ōkytokion*) in which Apollonius improved on the Archimedean estimates.²⁶ Eutocius does not elaborate, but Hero does. In the alternative estimate Archimedes is said to have established $211875 : 67441$ as a lower bound and $197888 : 62351$ as an upper bound on π . On first inspection these values are disappointing, for the former term is in fact an upper bound on π , while the latter, despite the large terms of the ratio, is a far poorer upper bound than $3\frac{1}{7}$. Thus, either Archimedes has blundered woefully in the execution of this computation, or else the text has suffered corruption. On the latter hypothesis several attempts at restoration have been made.²⁷ My own accepts the lower bound as given, but proposes that it has been mistakenly cited as a lower bound when it was actually derived as an upper bound; for one would not wish to suppose that a good value like $211875 : 67441$ arose merely by chance through a clumsily managed computation, while the accuracy is precisely of the expected order, about one part of the denominator.²⁸ Comparable accuracy for the lower bound suggests the value $197888 : 62991$, where the manuscript reading of 62351 is easily accountable through scribal error.²⁹ The correctness of the received numerator is indicated by the factorization $197888 = 2^8 \cdot 773$, where the factors of 2 correspond to the successive doubling of the number of sides of the bounding polygons. The ratios as emended can both be obtained through an Archimedean procedure commencing with decagons and ending with the 640-gons. But as in the *Dimension of the Circle*, the computed bounds could be reduced to lower terms with negligible loss of accuracy, yielding $333 : 106$ as lower bound and $377 : 120$ as upper bound, from the continued fractions $3 + \frac{1}{17 + \frac{1}{15}}$ and $3 + \frac{1}{17 + \frac{1}{17}}$, respectively. This at once suggests an alternative intermediate value $355 : 113 = 3 + \frac{1}{17} + \frac{1}{16}$. This same value was established by Chinese geometers in the 5th century A.D.; it was derived again by Western computers in the 16th century. Since it is accurate to about one part in twelve million, however, the ancient computation implied by Hero's figures could not have established it rigorously save through some essential modifications in procedure.³⁰

Since the value $3\frac{1}{7}$ suffices for the practical purposes of metrical geometry, ancient documentary evidence of such more refined computations is sparse. Only in astronomy was a closer estimate demanded. For the trigonometric studies by Hipparchus around the mid-2nd century B.C., Toomer has reconstructed the value $3438 : 21600$ as effective estimate for the ratio $1 : 2\pi$.³¹ Since this might be derived simply as the arithmetic mean of the inverses of Archimedes' values in the *Dimension of the Circle*, we cannot cite it in support of Hero's testimony to the refined Archimedean values. Indeed, the larger terms of the latter indicate a degree of accuracy about ten times that required for Hipparchus' uses. In the great system of mathematical astronomy compiled by Ptolemy three centuries later the value adopted for π is $3^\circ 8' 30''$ in the sexagesimal notation, that is, $\frac{377}{120}$.³² This is precisely the value we derived from the upper bound reported by Hero. Ptolemy provides no derivation of the value, but observes merely that

it lies between the Archimedean values $3\frac{1}{4}$ and $3\frac{1}{2}$. Although he might well have depended on refined computations such as that cited by Hero, an alternative account is equally possible on the basis of the trigonometric table in Ptolemy's Book I. There the value for chord 1° is $1^\circ 2' 50''$ for a circle of diameter 120° ; one might thus produce for π the estimate $360 \cdot \text{chord } 1^\circ : 120 = 3^\circ 8' 30'' : 1$.³³ Ptolemy sets out in full detail the computation leading to his value for chord 1° , and both this value and the estimate for π corresponding to it are correct to the nearest digit in the second sexagesimal place after the units. Thus, his results do not of themselves indicate prior knowledge of this computed estimate for π . This does not rule out that such knowledge might have helped steer the course of the computation of the chords. For instance, the value of chord 1° is found via the inequality $\frac{3}{4} \text{ chord } 1\frac{1}{2}^\circ < \text{chord } 1^\circ < \frac{1}{2} \text{ chord } \frac{3}{4}^\circ$, where successive bisections of the angle 24° have led to the results chord $1\frac{1}{2}^\circ = 1^\circ 34' 15''$ and chord $\frac{3}{4}^\circ = 0^\circ 47' 8''$. This leads to chord $1^\circ = 1^\circ 2' 50''$, but only because Ptolemy has chosen to adopt an upward rounding off for chord $\frac{3}{4}^\circ$ ($= 0^\circ 47' 7'' 24''' \dots$) instead of the expected downward rounding off. Had he instead adopted the lower figure, the value for chord 1° would become $1^\circ 2' 49''$ and give rise to a lower estimate for π (e.g., $3^\circ 8' 27''$). These difficulties are resolved by carrying through the computations to the third place after the units, as the commentator Theon shows.³⁴ But awareness in advance that $377 : 120$ was a good approximation for π could have served as a reliable guide.

The juxtaposition of Archimedes and Ptolemy is interesting in that it reveals the difference between the theoretical and the practical approaches to this computation. Ptolemy seeks good approximations for the practical needs of calculation. Expressing results to the nearest digit gives him the advantage of cancellation of errors, so that he obtains values rather closer than those attainable via a strict Archimedean procedure. Now his value $3^\circ 8' 30''$ turns out to be slightly greater than π , despite its association with the table of chords, that is, a derivation based on inscribed polygons. The Archimedean procedure, by contrast, controls the direction of rounding off, so that only lower bounds can result from the inscribed figures. Moreover, the production of both upper and lower bounds provides a direct indication of the degree of accuracy of the approximation. In this way, Archimedes reveals a greater concern for the theory of this computation than for the particular numbers it happens to yield. Thus, one cannot be satisfied by the view, already voiced in antiquity, that Archimedes' aim in this work was essentially practical.³⁵ This might indeed characterize the older Mesopotamian efforts, doubtless dependent on empirical measurements, which resulted in the value $3\frac{1}{4}$.³⁶ But it fails entirely to capture the subtlety of Archimedes' effort. His value of $3\frac{1}{2}$ is certainly more accurate than the older figure. But how is it more "practical" when one considers the manipulation of fractions in ordinary contexts? Furthermore, what is "practical" about his lower bound $3\frac{1}{4}$? If, then, Archimedes' purpose in the *Dimension of the Circle* was to deploy geometric theory for the presentation of an algorithm and then to illustrate it through a cleverly managed calculation, he would in effect be demonstrating the greater effectiveness of *theoretical* over purely practical procedures in geometry. But

this makes puzzling the motive behind the extended computation implied in the figures reported by Hero. For there is little if any theoretical insight to be gained merely by carrying out the same computational procedure to one or two more decimal orders of accuracy. Neither would this be expected to produce results of greater utility for practice. One thus has some cause to question Hero's assignment of these numbers to Archimedes, and to suspect that perhaps another geometer, like Apollonius in his *Okytokion*, was responsible for them. We shall return to this question later.³⁷

The results in the *Dimension of the Circle* cannot be viewed as attempts at a direct solution of the circle quadrature. But ancient and medieval commentators recognized that the proof of the area rule in the first proposition depends on the postulate that one can produce a straight line equal to the circumference of the circle.³⁸ In effect, then, Archimedes has not solved the circle quadrature, but rather revealed its equivalence to the problem of producing that straight line. Many of Archimedes' theorems have the same force: they demonstrate that the measurement of a specified curvilinear figure can be reduced to that of a certain circle. For instance, in the version of the *Dimension of the Circle* known to Hero, Archimedes proves that the area of a sector is one-half the product of its radius times its arc.³⁹ Presumably, the same work included the proof that the surface of a truncated right cone equals one-half its slant height times the circumference of its base, as one may see by rolling it out into a plane circular sector. In *Sphere and Cylinder* 1, 14 the same result is expressed in the form that the conical surface equals the circle whose radius is the mean proportional between the slant height and the radius of the base circle. Similarly, the principal result of that work proves that the surface of the sphere equals four times its greatest circle, while the area of any segment equals the circle whose radius equals the line drawn from the vertex of the segment to a point on its circular base.⁴⁰ In like manner, the volumes of conical and spherical segments are referred to the volumes of cylinders. In *Conoids and Spheroids* the area of the ellipse is shown to be equal to that of the circle whose diameter is the mean proportional between its major and minor axes; the volumes of conoids of revolution are expressed in terms of associated cones.⁴¹ In all these instances, then, the measurement of the figure has been reduced to the fundamental result of the measurement of the circle.

Two such quadratures appear in the *Book of Lemmas*, a work preserved in Arabic and whose translator Thābit ibn Qurra reports opinions of its Archimedean provenance.⁴² In the fourth proposition the figure "which Archimedes calls *arbelos* (leather cutter) is drawn, a trigram bounded by three semicircles tangent at the points A, D, C [Fig. 2(a)]. If BD is the perpendicular half-chord in the large semicircle; then the area of the circle of diameter BD equals the *arbelos* ADCBA. The proof substitutes the relation $BD^2 = AD \cdot DC$ into $AC^2 = AD^2 + DC^2 + 2 AD \cdot DC$ to obtain $BD^2 = \frac{1}{2} AC^2 - \frac{1}{2} AD^2 - \frac{1}{2} DC^2$; since circles have the ratio of the squares on their diameters, this establishes that the circle of diameter BD equals the difference between the semicircle of diameter AC and the two semicircles of diameters AD, DC, respectively, where

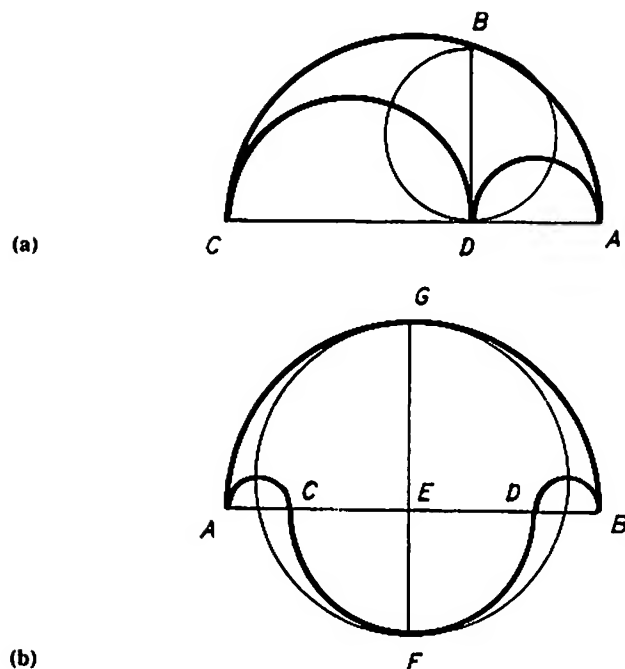


Figure 2

the latter difference is the *arbelos*. A similar quadrature appears in proposition 14: the *salinon* (salt cellar)⁴³ is bounded by the semicircle ABG, the two equal semicircles on the diameters AC and BD, and the semicircle DFC [see Fig. 2(b)]. By a proof like that for the *arbelos*, it is shown to equal the circle of diameter GF. One observes that the proofs for both figures are very much in the style of the lunule quadratures by Hippocrates of Chios. But there is a forward-looking aspect implicit in two additional lemmas to the figure of the *arbelos*. In Prop. 5 it is proved that the circles EFG, LMN are equal, where each is tangent to the large semicircle, to one or the other of the small semicircles, and to the line which separates them [see Fig. 3(a)]. In Prop. 6 the diameter EF of the circle drawn tangent to all three semicircles is $\frac{9}{10}$ diameter AC when $AD : DC = 3 : 2$; that is, $CP : PO : OA = 4 : 6 : 9$ [see Fig. 3(b)].⁴⁴ The latter result finds its natural extension in the "ancient proposition" which Pappus presents in Book IV of the *Collection*: if a succession of such tangent circles is inscribed in the *arbelos*, the elevation of the center of the first above the base diameter equals its own diameter, the elevation of the center of the second equals twice its own diameter, the elevation of the center of the third equals three times its own diameter, and so on [see Fig. 3(c)].⁴⁵ The theorem given by Pappus may in its turn be viewed as an application of special cases of the problem of constructing circles tangent to three given circles, the complete solution for which

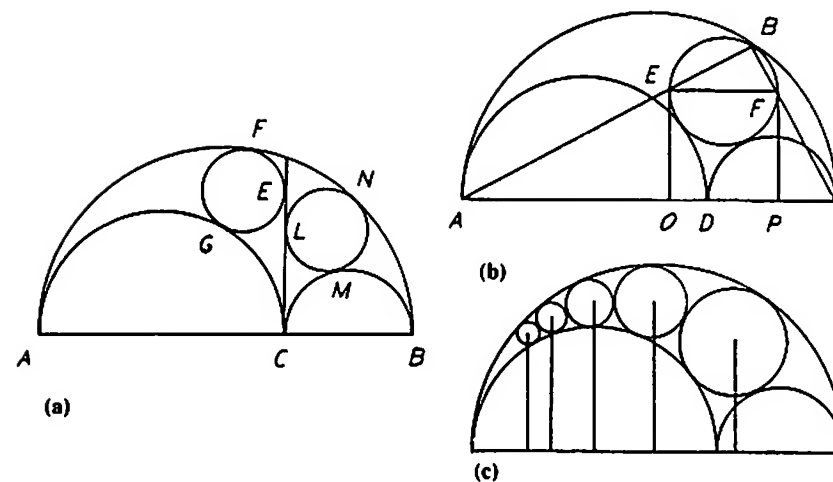


Figure 3

was worked out by Apollonius in his work *On Tangencies*. In this way one can perceive a link between studies initiated by Archimedes and an inquiry pursued later by Apollonius. We shall later discuss comparable links between their studies of circle measurement and *neuses*. Unfortunately, the confused state of the manuscript tradition of the *Lemmas* available to the Arabic translator prevents our delineating precisely those parts actually due to Archimedes. The quadratures of the two figures are examples, if quite simple ones, of the class of curvilinear measurements which dominate the known work of Archimedes. But the theorems on the tangent circles fall within a field of problems not otherwise represented in the Archimedean corpus, but quite characteristic of efforts in the generation after him. We might thus well prefer to assign these results to a follower of Archimedes, rather than to Archimedes himself.

The determination of the areas of figures bounded by spirals further illustrates Archimedes' methods of quadrature. The Archimedean plane spiral is traced out by a point moving uniformly along a line as that line rotates uniformly about one of its endpoints. The latter portion of the treatise *On Spiral Lines* is devoted to the proof that the area under the segment of the spiral equals one-third the corresponding circular sector. For instance, the area lying between the spiral and the radius vector BE is one-third the sector BEH; the area between the whole spiral and the terminal radius BA is one-third the whole circle of radius BA (see Fig. 4). The proofs are managed in full formal detail in accordance with the indirect method of limits. The spirals are bounded above and below by summations of narrow sectors converging to the same limit of one-third the entire enclosing sector, for the sectors follow the progression of the square integers.⁴⁶ This method remains standard to this very day for the evaluation of definite integrals as the limits of summations. It is applied throughout the *Conoids and Spheroids* as well for the measurement of the volumes of solids of revolution.

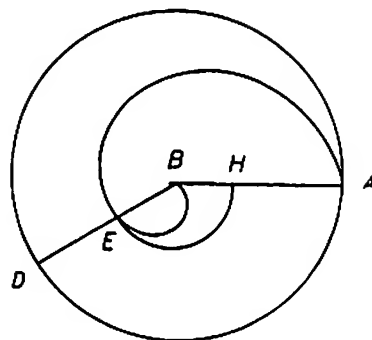


Figure 4

But Pappus reports an alternative treatment, "a remarkable strategy" (*thaumastê epibolê*), by which Archimedes effected the quadrature of the spiral.⁴⁷ As this occurs in the context of communications between Archimedes and Conon of Samos, the latter having been deceased for "many years" at the time of the writing of the extant *Spiral Lines*, Pappus here must be preserving for us a much earlier treatment of this theorem. As in the later method, Archimedes here introduces a set of inscribed sectors bounding the spiral from below [see Fig. 5(a)]. At the same time he considers the parallel slices in a cone, that is, the thin cylinders inscribed in it, whose sum bounds the cone from below [see Fig. 5(b)]. Since in both instances the constituent elements follow the progression of the square integers, it follows that their sums bear the same ratio to their respective containing figures. Thus, the lower bounding figure for the spiral has the same ratio to the whole circle that the lower bounding figure for the cone has to the cylinder. As a comparable proportion obtains for the circumscribed bounding figures, it follows that the spiral is to its circle as the cone is to its cylinder, namely, one-third. By a closely related technique, Pappus later establishes that the spherical spiral separates the surface of the hemisphere into two portions

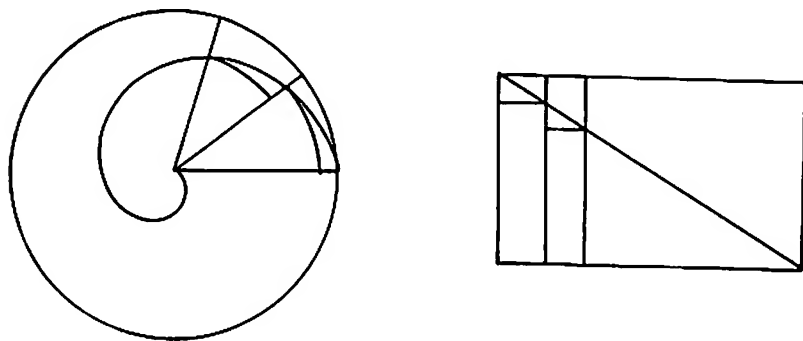


Figure 5

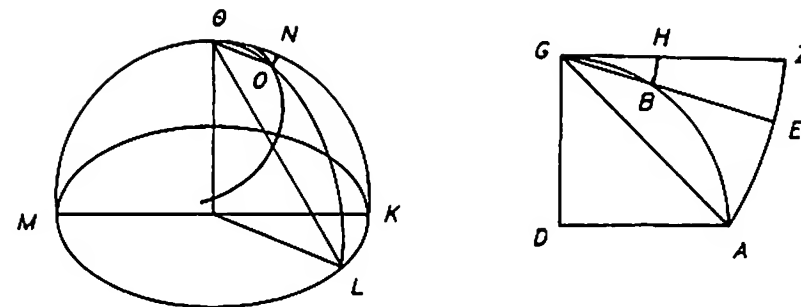


Figure 6

equal, respectively, to eight times the segment ABG and eight times the triangle ADG, where the sector ADGB is a quadrant of a great circle in the sphere (see Fig. 6).⁴⁸ As for the plane spiral, the surface of the figure bounded by the spherical spiral toward the initial position θNK is approximated above and below by surface elements corresponding to the elements in a second diagram, namely, the small sectors drawn to the segment ABG. This depends on Archimedes' theorem from *Sphere and Cylinder* I that the area of the surface element θNO is proportional to the square of θO , just as the area of the sector BHG is proportional to the square of BG, where $BG = \theta O$.⁴⁹ Hence, the sums have the same ratio to their respective enclosing figures. It follows (details of the limiting argument are omitted by Pappus) that the figure bounded by the spiral is to the hemisphere as the segment ABG is to the sector AGZ. The sectors AGZ and ABGD equal each other, however, while the latter is one-eighth the surface of the hemisphere.⁵⁰ Thus, as claimed, the figure bounded by the spiral is eight times the segment ABG. Taking the differences, we have that the other portion of the hemisphere bounded by the spiral equals eight times triangle ADG. The extremely close similarity in method used for evaluating the area of this form of the spiral to that of the plane spiral (as in Pappus) and its heavy dependence on results from Archimedes' *Sphere and Cylinder* encourage assigning it an Archimedean provenance. One may note that the spherical spiral is a striking example of a curvilinear figure, like the lunules of Hippocrates, found to be equal to a constructed rectilinear figure.

It is Archimedes' results on the tangents to the spirals which move toward the actual solution of the circle quadrature. Given the spiral whose initial ray is OB, the tangent drawn to any point C on the curve meets the line drawn from O perpendicular to the ray OC such that the intercept OE equals the arc CF in the circle of radius OC which subtends the angle COB (Fig. 7). Archimedes devotes the major portion of *Spiral Lines* to the formal demonstration of this result for the three cases of arcs equal to, less than, or greater than the whole circumference of the circle (Props. 18, 19, and 20, respectively), together with the requisite lemmas on the general properties of the curve and its tangents. The treatment is entirely formal in the synthetic manner, so that the underlying conception of

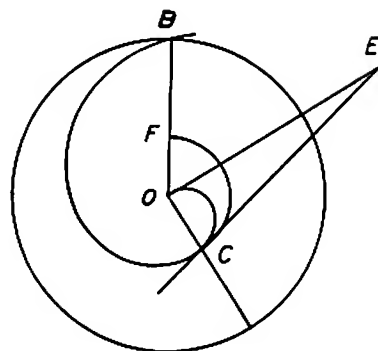


Figure 7

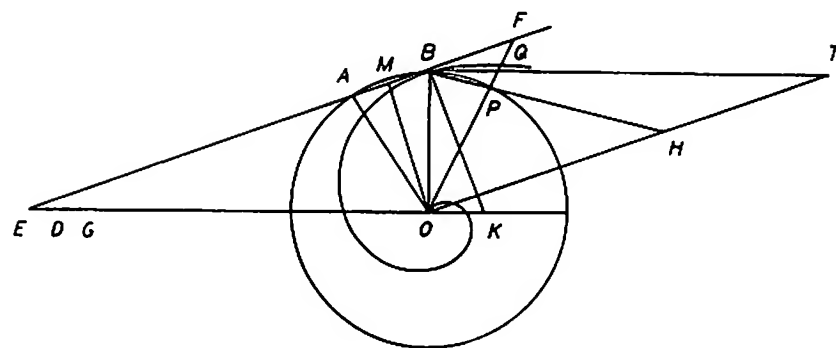


Figure 8

the argument is rather difficult to discern. I thus propose to reconstruct an analysis, closely modelled on the synthesis Archimedes gives, to reveal how its essential features emerged. One should then be able to work through the details of Archimedes' formal treatment with greater ease.⁵¹

We may frame this analysis as a problem: to draw the tangent to a point on the spiral, for example, the point completing its first full turn. This will be done by specifying the length of the subtangent OE where the tangent drawn to B meets the line drawn perpendicular to the ray OB (Fig. 8). Let a point Q be taken on the spiral on the forward side of B and let the ray OQ be drawn, meeting the circle of radius OB at P. By the definition of the spiral, the rays are in proportion to the corresponding angles, that is, to the arcs of this circle; thus $OQ : OB = \text{arc PB} + C : C$, where C equals the circumference of the circle of radius OB. Since $OB = OP$, we obtain $OQ - OB : OB = QP : OB = \text{arc PB} : C$. Now extend OQ to meet the tangent at F, and draw the line BP extended to meet in H the line drawn from O parallel to the tangent. Draw from B the line parallel to OE meeting OH extended in T. This makes OEBT a parallelogram, so

that $OE = BT$. Now by similar triangles, $FP : BP = OP : PH = OB : PH$. Since F is the point on the tangent corresponding to Q, $FP > QP$; since, also, $PB < \text{arc PB}$, it follows that $FP : BP > QP : \text{arc PB}$. From the above, $QP : \text{arc PB} = OB : C$. Thus, $OB : PH > OB : C$, or PH is less than C. Furthermore, PH is less than BT, since BT subtends the largest angle in the triangle BHT. Since $BT = OE$, it follows that PH is less than both OE and C. This does not establish a relation between OE and C as such. But it is evident that as P tends to B, Q and F converge, so that $FP : PB$ tends toward equality with $QP : \text{arc PB}$; that is, PH converges toward C from below. At the same time PH tends toward equality with BT. We thus have that the subtangent OE equals C, the circumference of the circle of radius OB, the common limiting values of the line PH. One might investigate in comparable fashion the relations arising for P situated on the rearward side of B. Moreover, the result generalizes for portions other than the full turn of the spiral. When the terminal ray OC does not coincide with the initial ray OB, the subtangent OE will be the length of the arc corresponding to the sector angle BOC in the circle of radius OC, where OE is perpendicular to OC (cf. Fig. 7).

Clearly, a treatment of this sort lacks the precision demanded of a formal exposition. But it reveals the length of the subtangent one requires for drawing the tangent, as well as the general line of the argument one might follow to obtain the formal synthesis. In particular, one perceives the role of the insertion (*neusis*) of a length shorter than the subtangent, that is, the line PH between the circle and the line OH parallel to the tangent.⁵² One might thus try to show that any line other than BE as drawn here will meet the spiral at another point besides B; that is, that if OD is less than OE, the line BD will cut the spiral at some point R on the rearward side of B, while if OD' is greater than OE, the line BD' will cut the spiral at Q on the forward side of B. This will establish that BE is the tangent, in accordance with the conception of the tangent in the ancient geometry.⁵³ Alternatively, one might consider the hypothesis that the tangent meets the perpendicular at E such that OE is not equal to C. The case where OE is greater than C ($= OG$) gives rise to the diagram we have been considering; for any point D lying between E and G, one can insert the length $PH = OD$ and so locate a point Q on the forward side of the spiral from B, where Q will be found to lie beyond F, the point on the extension of the assumed tangent EB; this contradicts the fact (proved in Prop. 13) that Q must lie below the tangent. Similarly, one may derive a contradiction from the assumption that the subtangent is less than C. This approach to the synthesis is in fact the one adopted by Archimedes in *Spiral Lines*, Prop. 18.⁵⁴

The informal analysis proposed here utilizes a somewhat dynamic conception of the limiting process in which as P approaches B, the triangle FPB becomes similar to BOE, while triangle OPH becomes congruent with OBT. To be sure, such conceptions are alien to the formal style of the ancient geometry. Yet one would not like to deny to Archimedes the availability of intuitions of this sort, especially in view of the importance of mechanical heuristic procedures for his other work. Moreover, it is difficult to imagine how one would approach the

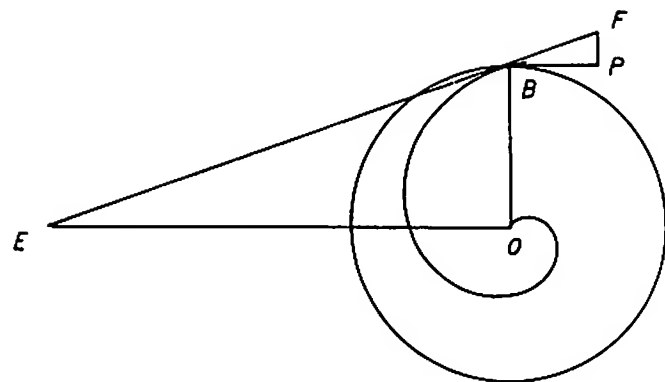


Figure 9

problem of the tangents to motion-generated curves like the spiral without some essentially kinematic intuition. One may note that the present form partly disguises the role of the differential triangle as one would introduce it in a more modern treatment. The triangle FPB would assume that position in the limit, where its side FP will be proportional to the radial velocity of the point tracing out the spiral as it arrives at B, while BP will be proportional to its velocity along the tangent to the circle at B (Fig. 9). Since the motions are uniform, the velocities will be proportional to the total distances traversed by B in these two senses, so that $FP : PB = OB : C$. But by similar triangles, $FP : PB = OB : OE$. Hence, $OE = C$. The procedure actually adopted by Archimedes is far removed from this simpler method, so that one must suppose that its conception of the tangent as the direction of the instantaneous motion composed from the motions which generate the curve, while fundamental for studies of tangents since the 17th century, was still unrecognized by the ancients.⁵⁵

The plane spiral is but one of several forms of spiral studied by the ancient geometers. We have already mentioned the spiral on the sphere, whose quadrature paralleled that of the plane spiral in Pappus' account. Pappus also reports of a construction relating the plane spiral to the spirals drawn to the cone and the cylinder.⁵⁶ Given the plane spiral BHA, if one erects $KH = HB$, perpendicular to the plane of the spiral, the point K will lie on the right cone of vertex B and axis BL (Fig. 10). Thus, as H traces the plane spiral, K will trace a "surface locus," the spiral on the cone.⁵⁷ If the ray LK is now extended to meet at θ the right cylindrical surface drawn over the circular base GDA, this point θ will trace another surface locus, the spiral on the cylinder. The latter was familiar to the ancients by the name *cochlias*. Apollonius devoted a writing to its study, while Archimedes is said to have invented the water-lifting device which shares both its name and its figure.⁵⁸ We should thus suppose that the ancient geometers discovered a variety of properties of these curves. It happens that the projective relationship just cited provides an alternative way to find the tangent to the plane spiral. First consider the cylindrical spiral: since the ratio $\theta D : \text{arc } DG$ is a

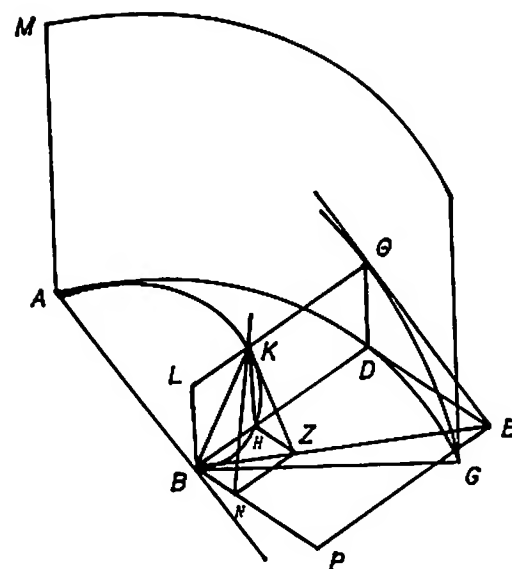


Figure 10

constant for all points θ , the curve may be viewed as the result of wrapping a plane triangle GAM against the cylinder,⁵⁹ so that the tangent to θ will be the line θE where DE is perpendicular to DB and equal to the arc DG . Since the cylindrical spiral is the radial projection of the conical spiral, the tangent to the latter at K will project onto the tangent to the former at θ via lines radiating from LB, and hence onto the line KZ , meeting the base plane in Z, such that $DE : HZ = \theta L : KL$. HZ will equal the arc of the circle of radius KL corresponding to the arc DG ($= DE$) of the circle of radius BD. Now the tangent to the conical spiral at K lies in the plane tangent to the cone at K, that is, to the plane through BK and perpendicular to the plane of BLK. This tangent plane meets the base plane in the line BP perpendicular to BH, while the tangent line to the conical spiral will meet BP in N, where ZN is parallel to HB. Since $HZ = BN$, BN will equal the arc of the circle of radius BH corresponding to the angle GBH. The plane spiral is the plane projection of the conical spiral, so that its tangent at H is the plane projection of the line KN, that is, the line HN. We may thus conclude that the tangent to the plane spiral at H meets the line BP drawn perpendicular to the ray BH such that the subtangent BN equals the arc of the circle of radius BH over the angle GBH. In this way, the conception of the spirals in the manner of surface loci has the potential to provide a heuristic for the results on the tangents to the plane spiral presented by Archimedes in *Spiral Lines*.

Did Archimedes view his findings on the spirals as a construction of the circle quadrature? A sign that he did seems to lie in his manner of expressing them: they are not given as *problems* in the drawing of tangents to spirals, but rather

as *theorems* on the properties of the tangents. Moreover, the later tradition did indeed include Archimedes' spirals, along with the quadratrix and the cylindrical spiral, among the known efforts at circle quadrature.⁶⁰ What, then, of its success? Certainly his followers did not consider the matter closed, for the studies of the quadratrix, for instance, appear to have been stimulated by Archimedes' efforts, not cut off by them. As for Archimedes' view, a remark from the preface to his *Quadrature of the Parabola* is suggestive:

Now of those who earlier applied themselves in geometry, some tried to prove [*graphein*: confirm by diagram] that it is possible to find a rectilinear space which is equal to the given circle and to the given segment of a circle, and after these things they attempted to square the space bounded by the section of the whole cone and a line by making assumptions not easily conceded, whence it was not acknowledged by the majority that these things had been discovered [*heuriskomena*] by them. But we understand that none of the earlier ones has tried to square the segment bounded by a line and a section of the right-angled cone [sc. a parabolic segment], which is what has now been discovered by us.⁶¹

At first it would seem unclear why Archimedes should view the successful quadrature of the parabolic segment as a distinctive achievement within this series of studies of curvilinear areas. For, after all, the lunules of Hippocrates and his own measurement of the spherical spiral, if the above view of its Archimedean origin and dating is accepted, would be familiar as instances of curvilinear figures found to be equal to certain rectilinear figures. Of the study of the ellipse (if that is what the obscure phrase "section of the whole cone" refers to)⁶² we know nothing other than the Archimedean quadrature given in *Conoids and Spheroids*. Still, there would appear to be a natural connection with the problem of circle quadrature. In this respect, the allusion to studies of segments of circles is of interest. For Hero preserves the full demonstration of the theorem that the segment of the circle is greater than four-thirds the triangle inscribed in it, that is, having the same base and altitude as the segment.⁶³ The treatment is in the same manner of Archimedes' *Dimension of the Circle*, and depends on the fact that the triangles AZB, BHG are together greater than one-fourth triangle ABG, where Z, H bisect the arcs AB, BG, respectively [Fig. 11(a)]; applying this result, in turn, to the triangles inscribed in the remaining segments of the circle, one obtains that the segment is greater than triangle ABG plus its fourth plus the fourth of its fourth and so on, where the sequence of parts converges to four-thirds. If one adapts this very procedure for a parabolic arc ABG and takes E, K to be the bisectors of the lines AD, DG, then one finds that the triangles AZB, BHG are together exactly one-fourth the triangle ABG, so that the summation yields that the parabolic segment is exactly four-thirds triangle ABG [Fig. 11(b)]. This is the manner of Archimedes' proof of this theorem in *Quadrature of the Parabola*, Props. 18–24.⁶⁴ Appreciating how an approximate quadrature of the circular segment so gives rise to an exact quadrature of the parabolic segment, Archimedes might well feel encouraged in the search for a circle quadrature.

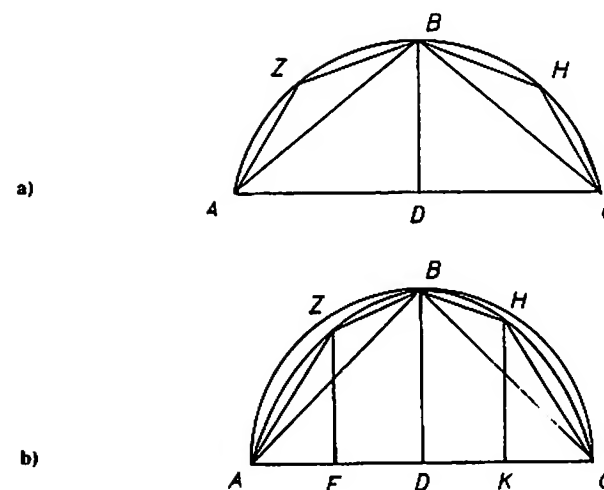


Figure 11

Here Archimedes says nothing about the spirals. But the view that he had already made his basic discoveries relating to these curves is supported by internal evidence. In the preface to *Spiral Lines*, communicated several years after *Quadrature of the Parabola*, Archimedes speaks of a list of theorems he had sent off to Conon years earlier. The promise to provide their formal proofs was being fulfilled in a writing already sent (*Sphere and Cylinder II*), together with the present work (*Spiral Lines*) and another yet to follow (*Conoids and Spheroids*). He states the theorems at issue, and among them are the principal results on the areas and the tangents of the spirals. Now Conon had died many years before the writing of the treatise *On Spiral Lines*; indeed, his recent death had been the occasion for Archimedes to address the *Quadrature of the Parabola* to the new correspondent Dositheus.⁶⁵ Thus, before this Archimedes had communicated to Conon some form of these results on spirals, doubtless in a manner like that preserved by Pappus.

It thus appears that Archimedes' solution of the circle quadrature via the tangents to the spiral might fall into that class of efforts "making assumptions not easily conceded," so that the problem remained open. This lends a certain poignance to a remark he makes in the preface to one of his latest writings, *The Method*, addressed to Eratosthenes:

It happens that these theorems [on the volumes of certain sections of cylinders] differ from those discovered earlier [on the volumes of conics of revolution]. For the latter figures, namely the conoids and the spheroids and their segments, were compared by us in magnitude to figures of cones and cylinders, but none of them was discovered to be equal to a solid figure bounded by planes. But each of those figures [newly communicated] bounded by two planes and surfaces of cylinders is discovered to equal one of those solid figures bounded by planes.⁶⁶

Despite an entire career of diligent and masterly effort, the quest for measurements of curvilinear plane and solid figures equal to rectilinear figures had produced but a handful of actual instances, while the circle quadrature itself remained as elusive as ever.

PROBLEM SOLVING VIA CONIC SECTIONS

In the extant Archimedean corpus, only *Sphere and Cylinder II* includes applications of the method of analysis and synthesis. It is devoted to the solution of problems dealing with spherical segments: e.g., the division of the sphere into segments whose surface or volume are in a given ratio; or the construction of segments equal to a given figure and similar to a second. These depend on the principal results of *Sphere and Cylinder I*: that the surface of the segment ABG (excluding its base) equals the circle of radius AB; and that the volume of the sector equals the cone whose base equals the surface of the segment and whose altitude equals the radius of the sphere (Fig. 12).⁶⁷ The form of the latter result brings to mind Archimedes' expression for the area of the circle in *Dimension of the Circle*, as equal to the triangle whose base is the circumference of the circle and whose altitude is its radius. Indeed, Archimedes himself notes this analogy elsewhere as significant for his discovery of the measurement of the sphere.⁶⁸

It is quite straightforward, then, to divide a given sphere into segments whose surfaces have a given ratio (*Sphere and Cylinder II*, 3). Since these segments have the ratio $AB^2 : AE^2 = BX : XE$, one need only divide the diameter of the sphere into segments having the given ratio and then divide the sphere into parts having these line segments as their respective altitudes. The problem of dividing the sphere into segments where the volumes are in the given ratio, however, turns out to be a major undertaking (Prop. 4). Here Archimedes makes use of an expression for the volume of the segment established in Prop. 2: if on the diameter extended, one marks off the point P such that $KD + DX : DX = PX : XB$ (for K the center and DB the diameter of the sphere) (Fig. 13), then the cone whose base is the circle of diameter AG and whose altitude is PX will equal the segment of the same base with vertex at B. If one now determines the analogous point L producing the cone LAG equal to the segment DAG, then the analysis of the problem of dividing the sphere reduces to that of determining the three points X, P, L such that (a) $PX : XL$ is the given ratio; for the cones are as their altitudes and by construction they are equal, respectively, to the segments of the sphere; (b) $KD + DX : DX = PX : XB$; and (c) $KB + BX : BX = LX : XD$, from the determinations of P, L as just stated. Through a considerable exercise in the manipulation of proportions, Archimedes reduces this in its turn to the problem of dividing BD at X such that $BD^2 : DX^2 = XZ : Z0$, where $ZB = BK$ and $BZ : Z0 = PL : LX$. Since $PX : XL$ is given and BK is the radius of the sphere, Z and 0 are given, so that only the position of X remains to be solved with reference to the derived condition. One perceives that it entails the solution of a third-order relation in DX ($= 3BK - ZX$). For the present

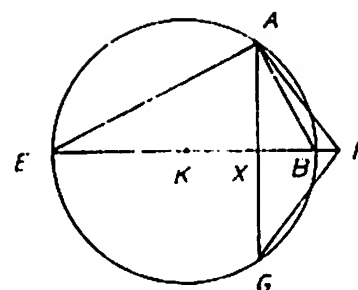


Figure 12

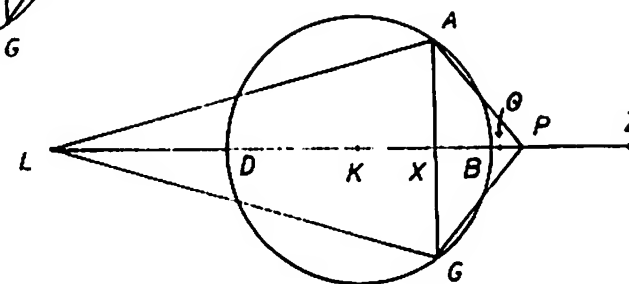


Figure 13

Archimedes merely assumes the solution of this problem for the purposes of producing the synthesis of the problem of dividing the sphere, but promises that "this shall be analyzed and synthesized at the end."⁶⁹

Through some textual mishap suffered by the work not long after its composition, the construction of this auxiliary problem became dislodged from the rest. Already by the close of the 3rd century *n.c.*, the geometers Diocles and Dionysodorus could find no such construction in the copies of *Sphere and Cylinder* available to them for study, and so each worked out his own form of the solution. These are reproduced by Eutocius, who reports that his own researches among the library shelves had brought to light an old book in deplorable condition which presented a treatment of these matters.⁷⁰ Since it was written in the Doric dialect, and as it referred to the conic sections by their archaic names ("section of the right-angled cone," rather than "parabola"), he deduced that this was an Archimedean work, for both features are characteristic of his writings. Eutocius goes on to present this text in a restored version devoid of archaisms and replete with appropriate citations of Apollonius' *Conics*. To this he adds versions of the methods used by Dionysodorus and by Diocles. We possess an alternative text of the latter, to which Eutocius' version is reasonably faithful (the principal discrepancy being that Eutocius presents not only the analysis, but also the synthesis, while Diocles omits the latter as "obvious").⁷¹ Thus, only the hypothesis of outright forgery on Eutocius' part could lead us to question that his version of the Archimedean method represents reliably the one originally promised by Archimedes. Not only are the motives for such a forgery entirely unclear, but also there is good cause to doubt that the expertise of a commentator like Eutocius was such as to produce an original treatment of comparable insight, yet so different from the alternatives which might serve for models.⁷²

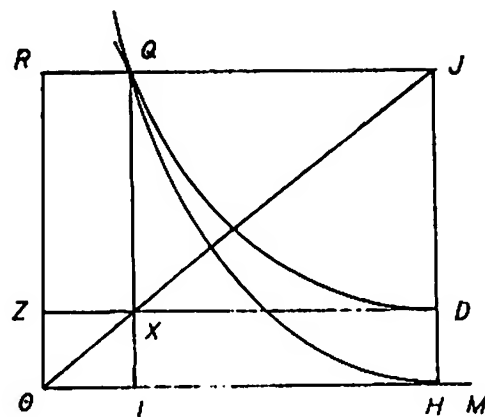


Figure 14

Let us sketch the analysis of Archimedes' solution of this problem. We are to divide the given lines BD at X such that $BD^2 : DX^2 = XZ : Z\theta$ for given points Z and θ . But it is now posed in the more general form in which the given area E may be any value (not necessarily BD^2), while the given line segment F is likewise arbitrary (not necessarily $Z\theta$). Assume then, that the given line ZD ($= \frac{1}{2}BD$) has been cut at X such that $ZX : F = E : XD^2$ (Fig. 14). Draw $Z\theta$ equal to F at right angles to ZD and complete the parallelogram θHJR with diagonal θXJ . We now determine M on θH extended by the relation $E = \theta H \cdot HM$. Thus, (a) $XZ : Z\theta = \theta H \cdot HM : XD^2$ by the relation for X , while (b) $XZ : Z\theta = \theta H : HJ$ by similar triangles, so that (c) $XD^2 = HM \cdot HJ$. If now we draw QI through X parallel to HJ , since $QI = HJ$ and $QJ = XD$, it follows that Q lies on a parabola with vertex at H , axis HJ and latus rectum HM (i.e., $QJ^2 = HM \cdot HJ$). Furthermore, from (b) $QI \cdot I\theta (= HJ \cdot XZ) = DH \cdot H\theta (= Z\theta \cdot H\theta)$, so that Q lies on the hyperbola passing through D with asymptotes $H\theta$, θZ . Since the parabola and the hyperbola are given, Q is given as their point of intersection.

This completes the analysis of the problem; the synthesis can be given merely by constructing the conics to determine Q and using this to establish that X does indeed divide ZD as to satisfy the desired relation. Now it is obvious that the original problem of dividing the sphere into segments of a given ratio is always capable of solution, for one or the other of the segments may take on values ranging from arbitrarily small to arbitrarily close to the whole sphere. Thus, to every ratio, however small or great, there will be a corresponding division of the sphere. But the auxiliary problem considered by Archimedes has been generalized, so that it is no longer obvious that a solution will exist for certain choices of E and F , the given area and the given line, respectively. This is apparent from the construction derived in the analysis, for one is not necessarily assured that the conics derived will indeed intersect within the specified range over ZD . Hence, a *diorism*, or determination of the conditions of solvability,

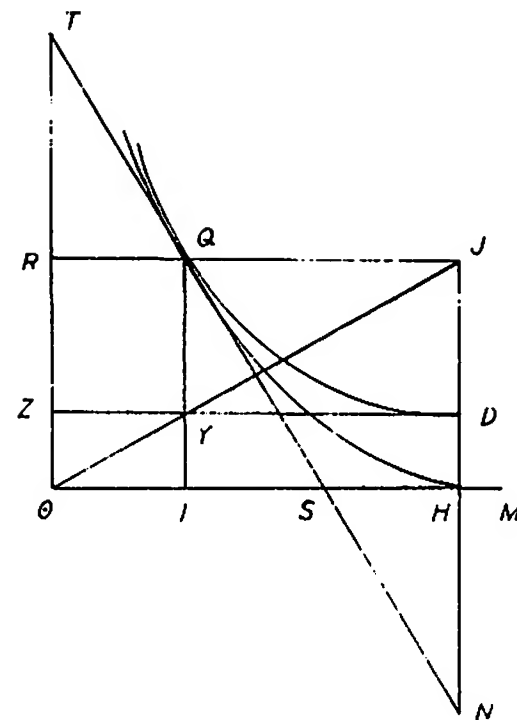


Figure 15

becomes necessary. Archimedes provides only the synthetic proof of the condition to be fulfilled: that the quantity $E \cdot F$ be less than $ZY \cdot YD^2$, where $2ZY = YD$. But it is not difficult to perceive the analysis which leads to this condition: the limit of solvability occurs where the hyperbola and the parabola have a common tangent at their intersection Q (Fig. 15). In the case of the parabola, this entails that $JH = HN$, N taken as the point where the tangent meets the axis JH extended¹³; hence, $QJ = 2HS$ (for S the point where the tangent meets θH), so that $DY = 2SI$. As for the hyperbola, the point of tangency bisects the segment of the tangent falling between the asymptotes, so that $TQ = QS$. Thus $\theta I = IS$, whence $DY = 2YZ$.¹⁴ Although Archimedes employs the tangents in the synthesis of this condition, his aim there is to prove that of all divisions X of the line DZ , this one yields a maximum for the value of $ZX \cdot XD^2$. It thus follows that if $E \cdot F$ is greater than $ZY \cdot YD^2$, the stipulated division of ZD will be impossible.

Archimedes' treatment of this problem is a nice example of how the process of generalizing stimulates the field of geometric studies by raising new questions for solution not posed in the special problems initially. We shall see several instances of this in the work of Apollonius, especially in his studies of *neuses* and plane loci. Moreover, his manner of investigating the normal lines to the conic sections is remarkably close to the Archimedean problem we have been

discussing here, including the use of intersecting conics to determine critical points, the properties of tangents to discover the limits of solvability, and even their elaboration in terms of maxima and minima.⁷⁵ Nevertheless, certain potentials in Archimedes' results here appear not to have been recognized for further development. Consider that Archimedes has shown how to draw conics to determine X such that $ZX : F = E : XD^2$ for any given line F , area E , and line ZD , where $ZX + XD = ZD$. That is, he has constructed the solution for a general class of cubic relations and determined the condition under which a solution exists for X within a given range. One might formulate other expressions of this type, ultimately to obtain an exhaustive classification of cubic relations together with the construction of their solutions under specified conditions of solvability. This is the very enterprise which Omar Khayyam undertakes in his algebra, and the method of solution he uses depends on intersecting conics.⁷⁶ As Eutocius' commentary on the *Sphere and Cylinder* was familiar among the Arabic geometers, we need not doubt that Archimedes' method in this problem could serve as a model for these studies. But if comparable efforts to generalize and classify these cubic constructions were taken up in antiquity, the extant record preserves no direct indication to that effect.

Indeed, without the model of Archimedes' solution to work from, Dionysodorus and Diocles prove quite insensitive to these possibilities of generalization. Their efforts to find solutions to the problem assumed without proof in their copies of Archimedes remain strictly inside the confines of the problem of dividing the sphere. Eutocius inaccurately describes Dionysodorus as unable to cope with Archimedes' form of the auxiliary problem, so that he was forced to derive an alternative reduction starting from the terms of the initial problem.¹⁷ This characterizes the method adopted by Diocles, but Dionysodorus actually follows the line of Archimedes' treatment quite closely. Although Eutocius reports only the synthesis, the analysis underlying Dionysodorus' procedure is not difficult to perceive. Assume that the given line ZD has been divided at X such that $XZ : ZO = BD^2 : DX^2$, where ZO and BD are also given (Fig. 16).

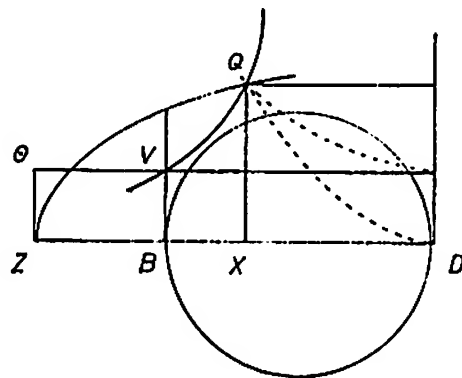


Figure 16

Since $XZ : Z\theta = XZ \cdot Z\theta : Z\theta^2$, if we set (a) $QX^2 = XZ \cdot Z\theta$, it follows that $QX^2 : Z\theta^2 = BD^2 : DX^2$, that is, (b) $QX : Z\theta = BD : DX$. The locus of Q that answers (a) is a parabola with vertex Z, axis ZD, and latus rectum Z θ ; that which answers (b) is a hyperbola with asymptotes ZD and the line perpendicular to it through D, such that the curve passes through the point V for $BV = Z\theta$. Hence, the intersection of these two conics determines that point Q leading to the required division of ZD at X. No diorism is provided; but as we have already noted, none is needed, since the restricted problem of division is always solvable. Although the actual curves derived by Dionysodorus differ from those introduced by Archimedes,⁷⁸ the close resemblance of the two methods to one another cannot be missed.

By contrast, Diocles returns to the initial conditions of the sphere division, specifying the points X, L, and P via three proportionalities. Here I present only the construction he derives in the analysis: assume as known the positions of L, X, and P in relation to the diameter DB of the sphere (Fig. 17).²⁹ Draw Dd at right angles to DB at D and of given length (namely, in the context of the restricted problem, one-half the diameter DB; but here Diocles allows for any value of the given line). Complete the rectangle defg with diagonal dXf, draw QXh parallel to ef, and extend eX to meet dg extended at j. Next determine k and m such that Dk = Bm = Dd, the given length, and draw through them lines perpendicular to DB extended to meet at r and s, respectively, the line

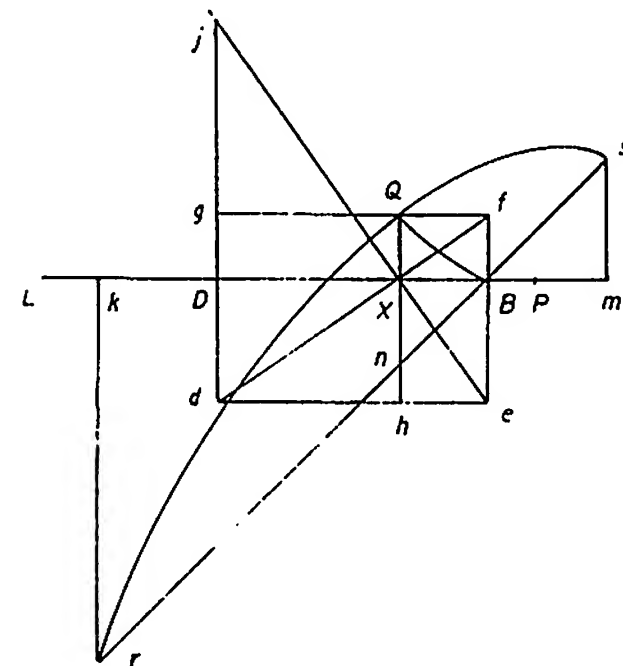


Figure 17

which makes half a right angle with DB at B. Let the line rBs met QXh at n. One finds that (a) $Qn^2 : m \cdot ns = PX : 2 XL$, where the ratio $PX : XL$ is given; and (b) $Qh \cdot hd = Be \cdot ed$. The former specifies Qn as the ordinate of an ellipse of diameter rs and whose latus rectum has to rs the ratio $PX : 2 XL$.⁸⁰ The latter specifies Qh as the ordinate of a hyperbola passing through B and having ed, dg as asymptotes. Since these curves are given, Q will be found as their intersection, and from Q one has the required point X dividing DB. Dioctes presents only the analysis of the problem; the synthetic demonstration is added by Eutocius.

Dioctes, like Dionysodorus, holds the narrow context of the problem of dividing the sphere, and so misses the opportunity for constructing the more general problem with its diorism in the manner of Archimedes. Yet we do well to note the essential similarities uniting these three presumably independent approaches. Doubtless, the procedure of applying intersecting conics for the solution of problems was relatively undeveloped at Archimedes' time, although Menaechmus' cube duplication was of course an established precedent for it. Neither Dionysodorus nor Dioctes can refer to Archimedes' solution, while Dioctes at least appears to come before Apollonius wrote the *Conics*. Their efforts thus reveal the maturity not only of the theory of the conics at the time near or before Apollonius, but also of the field of problem solving in which the conics were applied.

Another problem of this "solid" class, that is, entailing the use of conic sections for its solution, arises in connection with Archimedes' theorems on the tangents in *Spiral Lines*. As we have seen, the proofs of the determinations of the subtangents in Props. 18–20 assume certain points Q on the curve, and these are found via points P lying on the associated circle. The latter refer back to lemmas (Props. 5–9) in which P is found by inserting a given length PH between the circle and a given line. But how is this *neusis* to be effected? Apparently, Archimedes accepted it as obvious, by virtue of the very terms in which the operation is introduced; that is, it is the abstraction of what one may easily visualize in the form of the manipulation of a sliding ruler. But later geometers were not comfortable with this assumption on his part. Pappus reports a criticism to this effect:

It seems somehow to be no small error among geometers, whenever one solves a plane problem via conics or [higher] lines, and in general, whenever a problem is resolved [via curves] from a class not akin [to that of the problem]; for instance, the problem of the parabola in the fifth book of the *Conics* of Apollonius and the *neusis* of a solid toward a circle which is assumed by Archimedes in the book on the spiral; for without the intervention of any solid it is possible to find the theorem provided by him [sc. *Spiral Lines*, Prop. 18].⁸¹

One would suppose that the objection relates to the use of solid methods, that is, conics, for solving problems where the more elementary planar methods suffice. That is indeed the case for the Apollonian passage.⁸² In the Archimedean instance, however, the alternative constructions of the *neusis* via planar techniques either appeal to nonconstructive assumptions or require major modifi-

cations in the structure of Archimedes' proof.⁸³ Moreover, when Pappus goes on actually to "set out the analysis of the *neusis* assumed by Archimedes in the book on spirals, so that you might not be at a loss," the method he uses is "solid," for it requires conic sections.⁸⁴ While the text is in some disorder, it has been restored to make sense. It is in the form of two lemmas on the construction of the conics, leading to the solution of the *neusis*; in each case only an analysis of the construction, without the synthesis, is given.

The following paraphrase will show its relation both to *Spiral Lines* and to the solving methods used for the solid problem of *Sphere and Cylinder* II. It is required to insert a given length between a given circle and a given line as to pass through a given point on the circle.⁸⁵ Here the line is a diameter LN of the given circle of center O and radius OR; let the line PH be inserted between them, where PH has the given length θ and its extension passes through the given point B on the circle (Fig. 18). Since HB and HL are secants to the circle, $NH \cdot HL = PH \cdot HB$. If we draw KH perpendicular to LH such that (a) $KH = HB$, it follows that (b) $NH \cdot HL = PH \cdot KH$; since PH is given, the locus of K is a parabola, for if V is the intercept with the diameter ROR', then $OV \cdot \theta = NO \cdot OL = ON^2$, while $KH \cdot \theta = NH \cdot (NH + 2ON)$, so that $\theta \cdot (KH + OV) = \theta \cdot VW = OH^2 = KW^2$; that is, K is on the parabola of vertex V and axis VO and latus rectum θ . Furthermore, (a) sets K on a hyperbola of vertex B and diameter $BB' (= 2BZ)$ such that $KY^2 (= HZ^2) = BY \cdot YB'$. The intersection of these two conics thus produces the point K from which the required position of PH follows.⁸⁶

While this construction does not directly require the "intervention of a solid," as Pappus oddly expresses it, this doubtless is the "solid *neusis*" at the base of his objection to Archimedes' procedure.⁸⁷ We must then suppose yet another

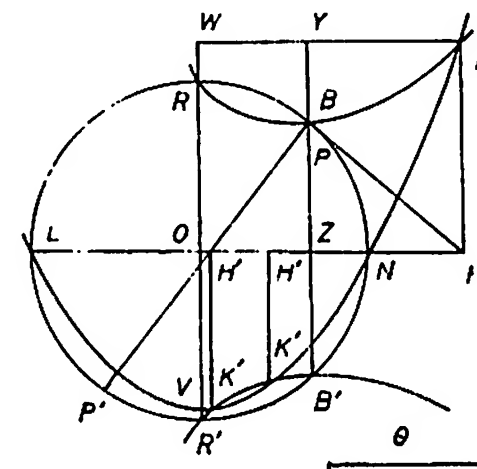


Figure 18

Now some accuse [Archimedes] of having needlessly employed a solid problem;...they show how it is possible also to find by means of planes the line equal to the circumference of the circle, applying the cited theorems on the spiral.²²

As for this method of effecting the *neusis* via conics, Archimedes was perfectly able to produce such a construction. But that he did not is clear from Pappus' need to supply it, filling a gap in the Archimedean presentation. Furthermore, Archimedes could hardly have expected his readers to assume such a construction of the *neusis* as obvious. It thus appears that he would not have admitted the force of the objection Pappus raises against the introduction of *neuses*. We shall see additional instances of this difference in attitude, as we turn next to other problem-solving efforts by Archimedes.

Arabic sources preserve for us two Archimedean problems which, like the problems in *Spiral Lines*, rely on *neuses* for their solution where alternative procedures making use of conics can be found, and indeed were found by later geometers. One of these is an angle trisection implied by a proposition in the *Book of Lemmas*, the work we consulted earlier for Archimedes' quadratures of the *arbelos* and the *salinon* figures. The second effects the inscription of a regular heptagon in a given circle. In both cases we are indebted to Thābit ibn Qurra for the Arabic translations which preserved knowledge of these results and stimulated a host of related efforts among Arabic geometers.

Let it be assumed that the heptagon has been inscribed in the given circle (Fig. 19), so that the side BH is the chord of an arc one-seventh of the circle, the diagonal AH is the chord of the double arc HLA, and the diagonals AB, BZ, HE are chords of triple arcs; let AB, HZ meet in C and HE, ZB meet in T and join CT. In the triangle DHB, the angles at H and B are equal, since they subtend the equal arcs BE, HA; thus lines HD and DB are equal. In the triangles HDC, BDT, the vertical angles at D are equal, while the angles at H and B are equal, since their associated arcs EZ, ZA are equal; thus the triangles are similar, and since HD = DB, they are also congruent. Hence, HC = TB. The quadrilateral HBTC thus has equal sides HC, TB and equal base angles at H and B; it may then be inscribed in a circle, so that angles CTH, CBH are equal (for they are subtended by the same chord CH). Now angle CBH is twice CAH; while DCH equals the sum of CHA, CAH, that is, twice CAH. Thus angle CBH equals each of the angles DCH, CTH. This makes the triangles CTH, DCH similar, so that $TH \cdot HD = HC^2$. Since $TH = CB$, $HD = DB$, and $HC = CA$, one has (a) $CB \cdot BD = AC^2$, a relation of the segments of line AB. Furthermore, since CHD and DAH are equal angles, triangles CHD and DAH are similar, so that $AD \cdot DC = DH^2$. Since $HD = DB$, it follows that (b) $AD \cdot DC = DB^2$, a second relation of the segments of AB. Thus the problem of inscribing the heptagon has reduced to that of finding two points C, D dividing the line AB according to the relations (a) and (b).⁹²

This derived problem is solved in Archimedes' preliminary lemma. It is there shown that if on the given line CB, one draws the square CBF₁G with diagonal BF₁, and the *neusis* of line AF is effected so that the triangles KFG, LCA are equal in area, then the segments of BA will satisfy the relations (a) and (b) (Fig. 20a). Again, only a synthesis is given. But one can construct an analysis which leads from these relations to the condition of the *neusis* in a reasonably straightforward manner.⁴¹ Let it be assumed that the line AB has been divided at C and

correspondence in content linking the ancient and modern theories of conics is due in no small part to the intense diligence with which 16th- and 17th-century geometers studied the works of Apollonius and Pappus.

⁹⁴ See above, notes 24 and 25.

⁹⁵ See Chapter 4. I have collected and discussed the testimonia on Aristaeus in my article, "On the Early History of the Conics."

⁹⁶ *Collection* (VII), II, p. 672

⁹⁷ Hypsicles in the preface to Book XIV of the *Elements*. See also Heath, *Euclid's Elements*, III, p. 512; and *History*, I, pp. 419 f. One may note that of the traditional fifteen books of the *Elements*, only the first thirteen are actually by Euclid. The fourteenth is by Hypsicles; the fifteenth is of uncertain authorship, but parts of it are by a 6th-century disciple of Isidore of Miletus.

⁹⁸ See K. Vogel, "Aristaeus" in the *Dictionary of Scientific Biography*. In the article cited in note 95 I suggest that this second Aristaeus may have been the father of Hypsicles, for their relation to the effort by Apollonius on the regular solids would appear to be of the same sort.

⁹⁹ See notes 26 and 33.

¹⁰⁰ Pappus, *Collection* (IV), I, pp. 282–284. The construction is paraphrased by Heath, who mistakenly makes the latus rectum $\sqrt{3}$ AH instead of its triple (*History*, I, pp. 241–243). The correct reading is adopted by Hultsch and Ver Eecke.

¹⁰¹ See Chapter 4.

¹⁰² *Collection* (VII), II, pp. 1004–1014.

¹⁰³ On the early expressions for the conics, see Chapter 4 and note 63.

¹⁰⁴ *Collection* (VII), II, p. 672.

¹⁰⁵ *Ibid.*, (IV), I, pp. 298–302. See also my article, "Archimedes' *Neusis*-Constructions in *Spiral Lines*," and Chapter 5(ii) above.

¹⁰⁶ See note 75 above.

¹⁰⁷ See note 39 above.

¹⁰⁸ Dozens of times throughout his commentary on the *Conics*, Eutocius explicitly cites the "scholia in the copies" on which he depends for alternative proofs and other explicatory material. One thus has little doubt that Pappus, too, depended on such manuscripts with annotations for the lemmas he provides to the *Conics* and other analytic works in *Collection* VII. Furthermore, one would expect that annotations in a work like the *Solid Loci* would reflect the significance of Apollonius' form of the theory of conics among later geometers. Thus, a source of this nature could well account for the major portion of the solid problems presented by Pappus.

¹⁰⁹ I present in the sequel the texts, with discussion, of the methods of al-Sijzi and al-Qūhī.

¹¹⁰ I elaborate the view that Anthemius' study is somehow based on pre-Dioclean sources in the article cited in note 80 above.

¹¹¹ See note 91.

¹¹² *Collection* (VI), II, Props. 53–54. We have mentioned this section in connection with Euclid's *Porisms* in Chapter 4.

¹¹³ *Conics*, preface to Book I (ed. Heiberg, I, p. 4).

CHAPTER 8

Appraisal of the Analytic Field in Antiquity

It is generally supposed that after the time of Apollonius the study of geometry entered a period of decline, interrupted only briefly by the work of Pappus in the 4th century A.D. This view agrees with the single most obvious and significant fact that the later tradition of geometry never again witnessed the rise of a first-rate mathematical mind comparable to Archimedes or Apollonius. On other counts, however, the view is questionable. The assumption, for instance, that "Pappus stands out as an accomplished and versatile mathematician" has never been subjected to scrutiny and is not likely to survive it.¹ More importantly, the view implicitly assumes that the later figures sought to make original contributions, but merely failed through technical incompetence. Until we obtain a better sense of the methods and motives of these figures, we shall thus be barred from an adequate assessment of their achievement in relation to that of their precursors.

We may usefully distinguish three aspects of their efforts: technical, textual, and evaluative. The technical competence of later writers like Hero, Theon, and Eutocius to expound mathematical texts, add minor lemmas or appropriate citations to the standard textbooks, and so on is manifest throughout their works; equally clear is the absence of original findings on their part, or even the intent to produce such. By contrast, the work of Pappus, in particular his *Collection*, contains such a rich diversity of more advanced materials that one naturally tends to assume his originality, save where he explicitly acknowledges his dependence on a prior source. The rigorous testing of this hypothesis would entail a detailed examination of all parts of the *Collection*, and this evidently overreaches the scope of the present context. A start is made in the sequel with the discussion

of his treatments of the geometric problems, and this reveals that his effort is not essentially different from that of any other commentator in this later tradition. If Pappus might on occasion submit a result as a finding of his own, as in the case of his method of cube duplication, we can perceive the clear dependence on earlier methods which effectively undercuts his claim. His extensive review of the works on analysis in Book VII only rarely rises above the level of what one might expect to find in the margins of annotated copies of these treatises, precisely the source exploited by Eutocius for the production of his commentary on Apollonius.² The more interesting parts of Book VII usually bear signs of a provenance in sources; for instance, the enunciation of the rule for measuring solids via centers of gravity is related to the theorems "in the 12th book of these *Elements*," and so appears to betray its placement within a commentary on Euclid and, as a comparison with a passage from Hero has indicated, its origination with Dionysodorus late in the 3rd century B.C.³ It is especially significant that the entire analytic corpus surveyed in Book VII derives from 3rd-century geometers, primarily Apollonius and Euclid. Surely one must infer that the problem-solving activity embodied in those works received no appreciable advancement at the hands of later geometers.

The decreased attention to advanced geometry beginning in the 2nd century B.C. should not be taken as a decline of the mathematical field in general, but rather a shift of interest from the special field of problem solving toward others, especially spherics, trigonometry, and numerical methods, which bore directly on the development of geometric and computational astronomy. Such shifts in the fashion of research, as it were, are a recurrent phenomenon in the history of science and mathematics, and indeed, changes of this sort had happened at least twice before within the ancient geometry. The study of irrational lines, for instance, was a prominent concern among geometers in the pre-Euclidean period; it engaged the efforts of superior geometers like Theaetetus and Eudoxus, while the formal synthesis of the classification of the irrationals constitutes the largest and most tightly structured part of Euclid's *Elements*.⁴ Yet we learn of no work in this field among later geometers beyond a study of "unordered irrationals" by Apollonius and a scattering of isolated results in Pappus which lie entirely within the Euclidean framework.⁵ Similarly, Eudoxus' limiting methods were the chief inspiration for the work of Archimedes and of necessity remain an element in the study of the circle quadrature and related problems of curvilinear measurement. Nevertheless, their significance to geometers other than Archimedes was minimal, as one appreciates most strikingly in their utter absence from the works of Apollonius.⁶ Doubtless, in the cases both of the study of irrationals and of the application of the Eudoxean techniques, the prospect of new and interesting discoveries was discouraging, so that later geometers largely abandoned their study for other fields. In much the same way, the field of analysis must have made the impression of having reached saturation through the accomplishments of Apollonius and his colleagues.

Thus, the later writers on geometry are not likely to be of great interest in connection with the development of technical methods or the discovery of new

results. They are extremely important, however, with respect to the issue of the textual transmission of these technical materials. The editorial tradition of collecting and commenting on interesting geometric results extends at least as far back as Hero, Geminus, and Menelaus in the 1st century A.D. and continues almost without interruption through late antiquity into the circle of Arabic scholars in the 9th and 10th centuries. Theon and Eutocius contributed critically to the preparation of editions of the works of Euclid, Archimedes, and Apollonius.⁷ It is thus through these commentators that we receive the overwhelming majority of our data relating to the ancient geometry. Understanding their editorial objectives and methods is thus indispensable not only for a view of geometric activity in later antiquity, but for one of the earlier researches as well. The detailed comparison of alternative texts which such an inquiry demands will be undertaken in the sequel to the present volume.

A third aspect of potential interest in these later writings lies in their efforts to evaluate the geometric achievement of their own and earlier times. The commentators typically volunteer observations on general matters dealing with the structure and methods of the field. In effect, they preserve a shadow of ancient metamathematical thinking, touching on such issues as the classification of problems, the relative status of the different constructing methods, the nature of the distinction between theorems and problems, and the role of the method of analysis. From their remarks we may hope to determine their positions on two questions bearing on the special problems of construction which have been our continuing interest in the present study: What precisely were the conditions to be imposed on a proper solution of these problems? Did the ancient search for solutions attain its goal? As we turn now to a survey of these matters, we may anticipate that the ancients' assessments differed in significant ways from those one would expect in a modern account of geometric constructions.

THE ANCIENT CLASSIFICATIONS OF PROBLEMS

In two passages preliminary to separate discussions of the special problems, the one (A) devoted to the cube duplication, the other (B) to the angle trisection, Pappus sets out in virtually identical terms the classification of geometric problems which one now generally accepts as the standard view among the ancients.⁸

The ancients (B: We) say that there are three kinds of problems in geometry: some of them are called "planar," others "solid," and others "linear." Now those which are capable of being solved (*lyein*) by means of straight line and circular arc would quite reasonably be called "planar"; for the lines by means of which such problems are found (*heuriskein*) have their genesis in a plane. But those problems which are solved when there is assumed toward their discovery (*heuresis*) one or several of the sections of the cone are called "solid"; for their construction (*kataskeuē*) necessarily employs surfaces of solid figures, namely the conic surfaces. Yet a third kind of problems is left, the one called "linear"; for there are taken for their construction lines different from those just mentioned, having a more diverse and rather contrived genesis (B adds: being generated from less regular surfaces and intertwined motions. Such are the lines found in the so-called "loci

with respect to surfaces'' ... and those found from the intertwining of plectoid surfaces and all manner of other surfaces and having many wondrous (*thaumasta*) properties about them). Such happen to be (*B*: Of the same kind are) the spirals and quadratrices and conchoids and cissoids (*A* adds: having many marvelous (*paradoxa*) properties about them).¹⁰ Since there is such a differentiation of the problems, the ancient (*B*: former) geometers were not able to construct (*B*: find) the above-mentioned problem of the two mean lines (*B*: of the angle) when they adhered to the geometric manner (*logos*) (*B*: when they sought it by means of the planes), for it is by nature solid.¹¹

Pappus recognizes an exactly analogous division of loci, as he explains in a remark prefacing his commentary on Apollonius' *Plane Loci*:

Those loci which concern us here are called "planar" and in general are such as are straight lines and curves (!) or circles. But solid loci are such as are sections of cones—parabolas or ellipses or hyperbolas. Loci are called "linear" such as are lines other than straight lines or circles or any of the cited conic sections.¹²

This classification scheme is of course not original to Pappus. He himself ascribes it to "the ancients," while his designation of the problem of cube duplication as "solid" is confirmed much earlier by Hero.¹³ As for the loci, Apollonius composed a work *On Planar Loci* which does indeed examine locus problems solvable via straight lines or circles. If the Aristaeus who compiled the treatise *On Solid Loci* was in fact a contemporary of Apollonius, as I have argued in Chapter 7, then this work would surely have dealt with those loci solvable as conics; for Apollonius describes the third book of his own *Conics* as containing "many remarkable theorems useful for the syntheses of the solid loci."¹⁴

Commenting on the last-mentioned passage, however, Eutocius presents a somewhat different view of the division of the loci:

It was customary for the ancient geometers to speak of "planar loci" whenever in the case of problems the problem comes about not only from a single point, but from several.¹⁵ ... Such as these are called "planar loci." But the loci called "solid" take their name from the fact that the lines by means of which the problems corresponding to these loci are drawn have their genesis from the section of solids, in the manner of the sections of the cone and many others. There are also others, called "loci with respect to a surface," which have their name from the special property relating to them.¹⁶

In this instance, Eutocius is clearly drawing from a different body of information and perhaps inferior to that accessible to Pappus two centuries earlier. Yet he appears to maintain a more primitive view than Pappus, one bringing together all cases generated from solids, rather than differentiating the "solid" from the "linear" type. Both writers call attention to the "loci with respect to surfaces;" for Pappus these fall within the "linear" category, while for Eutocius they would appear to constitute a class of their own. Now, Euclid wrote a work *On Surface Loci*, and it was seen in Chapter 4 to be connected not only with certain locus problems involving conics, but also with certain space curves, including forms of spirals, and the plectoidal curve which projects onto the quadratrix in the

plane. One would thus suppose that the wide variety of cases introduced in this Euclidean work came to be viewed as a class of loci additional to those loci connected with the constructions in the *Elements*, and only later was this new class reconsidered within the tripartite system known to Pappus. The same thing appears to have happened to the "loci with respect to means" arising from the work of Eratosthenes discussed in Chapter 6; for Pappus himself treats these loci as a class separate from his basic three, yet case by case assignable to one or other of them.¹⁷

Surprisingly, Proclus more resembles Eutocius than Pappus in his view of the division of loci:

I designate as "theorems of locus" those in which the same property obtains over some entire locus, and as "locus" (*topos*) the placement (*thesis*) of a line or a surface which makes one and the same property. For of the locus theorems some are framed with respect to lines, others with respect to surfaces. And since some lines are planar, others solid—planar are those whose conception in a plane is simple, such as that of the straight line; while solid are those whose genesis arises from some section of a solid figure, such as of the cylindrical spiral and the conic sections—I would then say that of the locus theorems with respect to lines, some have a planar locus, others a solid locus.¹⁸

In admitting both conics and spirals into his "solid" category, Proclus agrees with Eutocius rather than with Pappus. Yet in speaking of these as forms of "line locus" he differs from both, since they would term these "surface loci." Proclus goes on to report, on the authority of Geminus, a view of Chrysippus: that he likened the class of locus theorems to the "ideas" by virtue of their bringing together an infinity of cases answering to the specified terms.¹⁹ It is reasonable to infer that the more flexible classification of loci shared by Proclus and Eutocius also has standing in the older sources, and that the tighter view proposed by Pappus need not express the consensus of the ancient geometric field.

What of the provenance of Pappus' view? Clearly it would be impossible before the 3rd century B.C., since distinguishing the "solid" class must indicate a substantial development of the theory of the conic sections. One may detect from Pappus' passage on the loci that he is not reporting a very ancient view there, for he inserts it to elucidate a distinction of quite a different sort drawn from Apollonius' *Planar Loci*.²⁰ Apollonius is said to denote loci as "ephectic," "diexodic" and "anastrophic," according to whether the locus of a point, line, etc. is respectively of its same dimension or one or two dimensions higher.²¹ Pappus then indicates that planar, solid, and linear loci in his own sense can have one or another of these designations. It would appear that in his work restricted to planar loci, Apollonius did not have to remark on the special nature of other kinds of locus, like the solid class. This suggests that Pappus' long accounts of the division of the problems (passages A and B above) might include his own editorial elaborations of notions only partly prefigured in his sources.²² Indeed, one may well suspect that his rationale for their naming lacks any real authority from the older tradition, for it is essentially incoherent. How can one

presume to account for the "planar" class through its dependence on lines which are formed in a plane, when the same is no less true of the conic lines, the plane spiral, the quadratrix, the conchoid, and so on? Similarly, the "solid" class employs only the sections of the cone, indeed only its planar sections, rather than the full range of planar and space curves formed by the sectioning of the cylinder, cone, sphere, torus, and other solids.²³ Pappus himself elsewhere insists that the solid class is *not* to be understood through such a reference to solids, but only to the conic sections.²⁴ Discrepancies like this indicate that neither Pappus nor his sources had a clear rationalization for these designations.

If Pappus' view is thus a faithful reflection of the original sense of this classification, we would have to suppose that at the time when the field of problems had been sufficiently advanced so that geometers wished to distinguish between those solvable by means of straight lines and circles and those solved by means of conic sections, they seized upon the salient difference in the manner of generating these curves, the former lying in the plane, the latter arising through the sectioning of solids, and so named the respective classes "planar" and "solid," without considering the inconsistencies entailed. Extant evidence cannot deny the possibility of this view. But one can hardly be blamed for seeking a more satisfying alternative. Now, the terms "planar locus" and "solid locus," in the sense used by Pappus, are attested with Apollonius, but not earlier.²⁵ If they were coined by him, or very near his time, it is surely possible that his reasons for the choice were more profound than those just given. Now, in the case of problems, whenever a line segment is to be found in answer to a second-order relation, the construction can be effected by means of circles and straight lines, as one knows from the older techniques of the application of areas. Furthermore, when the segment is specified via a third-order relation, its construction may be effected via the intersection of conics; we have seen an important example of this in Archimedes' solution of the division of the sphere, while the cube duplication of Menaechmus is an older and more obvious instance.²⁶ Working out the details of this claim relating to the solid constructions would demand only the investigation of a number of additional cases, the type of inquiry characteristic of the treatises in the Euclidean and Apollonian tradition of analysis. Under this alternative view, it would be an intuition of the algebraic structure of the problems, as it were, which motivated the selection of terminology. Many of the cases taken up in Apollonius' *Plane Loci* seem quite remote from geometric interest, yet are easily appreciated as resulting through the urge to set out the variety of forms which give rise to solving lines and circles.²⁷ This view need hardly press us to suppose that the ancients attained, or even sought, a complete reduction of the field of problems to such algebraic forms. But it provides a sense in which an ostensibly arbitrary classification of problems, one whose rationale evaded the later commentators, might have originally been based on a real insight into the nature of these questions.²⁸

Terms like Euclid's "locus with respect to surfaces" and "locus with respect to three and four lines" and Eratosthenes' "locus with respect to means," for instance, are purely *descriptive* in their intent. In particular, they distinguish

certain classes of problems from others by virtue of the conditions which define the loci, rather than by considerations of the methods used in their construction. The terms "planar locus" and "solid locus" thus differ from the others in that they do seek a distinction on the basis of the construction method. Even so, their introduction need not at first have been intended for other than descriptive purposes. With Pappus, however, the distinction obtains a *normative* aspect:

The following sort of thing somehow appears to be no small error to geometers: whenever a planar problem is found by someone via conic or linear [lines], and on the whole whenever [some problem] is solved from a class other than its own, for instance, the problem on the parabola in the fifth book of Apollonius' *Conics* and the solid *neusis* toward a circle assumed by Archimedes in the book *On the Spiral*, for by using no solid one is able to find the theorem proved by him.²⁹

This rather timid pronouncement is the only ancient statement I know of which articulates the formal requirement to seek planar constructions in preference to others. Yet it is almost invariably presented in modern accounts as the principal objective of problem solving throughout the ancient tradition.³⁰ Is there evidence that the ancients ever actually subscribed to this rule? Pappus' remark indicates at once that neither Archimedes nor Apollonius felt bound by it, for their own constructions are singled out for criticism. One can safely assume that the suitable alternative constructions could have been produced, had either geometer admitted the need.³¹ The remark relative to Archimedes is especially confusing, first in that it appears to misconstrue what a "solid" construction is, and second in that the alternative construction later given by Pappus employs conics, and thus is itself of the solid kind.³² Surely Pappus meant to object to the appeal to the *neusis*, on the grounds that when its replacement by a solid construction is possible, as here, that would be preferable. But then, this would not illustrate Pappus' chief point about maintaining the appropriate boundaries between the planar, solid, and linear classes. Either Pappus is following an addled source, or else he is betraying a certain confusion on his own part. Pappus' responsibility for this passage is suggested by the fact that it is an insertion into his general account of the classification of the problems, but only in one of its appearances (text *B* cited above), not the other (*A*).

A review of the solutions devised for problems reveals little if any concern over this formal issue. Hippocrates constructs his third lunule by means of a *neusis*, with not the slightest indication that a planar method (which can be given here) is to be supplied.³³ In the 4th century geometers attacked the problem of the cube duplication with a host of different techniques, including the intersection of solids, the construction of special curves, and the use of mechanical motions.³⁴ As befits an early stage in the technical development of the field, the ambition is to find solutions by whatever means one can, to conceive of new methods and explore their potential freely, but hardly to restrict artificially the domains within which one or another method is deemed valid. To be sure, Euclid's *Elements* investigates only one segment of the field of problems, those constructible by planar methods. But that neither exhausted the planar field, as we become

fully aware through the investigations of Apollonius, nor did it deter Euclid from pursuing extensive studies of solid problems and the theory of conics.³³ The successful discovery of a solid construction for the cube duplication did not discourage geometers after Menaechmus from the search for alternative solutions of the linear kind. When Nicomedes castigates Eratosthenes over his method of cube duplication, it is not for reasons of formal correctness, but rather for practical feasibility.³⁴ These 3rd-century geometers share an interest in finding new mechanical procedures for solving geometric problems, and this seems to distinguish them from the geometers of the preceding century; but the formal ideal voiced by Pappus does not concern them.

The class of *neusis* constructions undergoes a notable transition during the 3rd century. In the work of Archimedes they are used in the study of spirals, as auxiliary to the formal proofs of the properties of the tangents to these curves. Given that Archimedes is often sensitive to matters of formal procedure, as one sees in his delicate management of the introduction of his mechanical method in his *Quadrature of the Parabola* and the *Method*, this surely attests to his acceptance of *neuses* as formally proper. Elsewhere he solves the problems of the inscription of the heptagon and the trisection of the angle via *neuses*, and geometers in the generation after him follow his example in devising *neuses* for the cube duplication and defining curves like the conchoids via *neusis* conditions.³⁷ But with Apollonius a new element enters. Although he himself is credited with an alternative construction of the cube duplication via *neusis*, he is also said to have used a method via conics for solving this problem. His *On Neuses* takes up only cases of the construction which may be effected by planar methods. Among his other analytic works, a division between planar and solid methods is indeed in evidence. The writings *On Planar Loci* and *On Tangencies* deal only with planar constructions, while others, like the *Section of an Area*, the *Section of a Ratio*, and the *Determinate Loci*, seem to have their principal role within the study of problems on the construction of conics.³⁸ It is thus clear that around Apollonius' time the field of geometry had matured to the effect that it had become important to regularize the formal methods of problem solving. The investigation of planar and solid problems as distinct classes would carry the expectation of discovering in detail the relative ranges of these methods, as would the effort to reduce other kinds of constructions, like the *neuses*, to one or the other class.

In this way the relative ordering of the different kinds of problems served to guide the research efforts of Apollonius and his contemporaries.³⁹ Other methods, such as the generation of special curves via motions, would continue to be of interest, since problems like the circle quadrature and the arbitrary division of the angle remain outside the scope of both the planar and the solid methods. But even with this, it is not clear that Pappus' rule of method had attained formal recognition, for we may note that Pappus himself does not abide by it. For instance, having prefaced his section on methods of cube duplication with the statement about the classification of problems (A), he goes on to present four different solutions: two are mechanical (the methods of Eratosthenes and of

Hero) and two are linear (those of Nicomedes and of Pappus himself).⁴⁰ Yet Pappus quite conspicuously informs us that the problem is of the solid type.⁴¹ Apparently, neither he nor Hero, who also states the solid nature of this problem, senses that in presenting solutions other than solid ones he has committed a formal error.

There is a much deeper difficulty, however: in the context of the Greek geometry the formal rule proposed by Pappus and the classification scheme corresponding to it are unworkable. If one can produce a planar construction for a problem, that does indeed establish the planar character of the problem. But the discovery of a solid construction for a problem does not yet suffice for securing the solid classification of this problem; one must go on to demonstrate that no planar construction is possible. Although comparable impossibility proofs were an essential feature of the ancient theory of irrational lines, we have no grounds for supposing that the Greek geometers ever did or could set forth the kind of proofs required for the classification of problems. The forms ultimately worked out in the 19th century depend on algebraic techniques of a sort alien to the Greek geometric approach.⁴² Despite this, Pappus and Hero seem perfectly comfortable in asserting the solid nature of the problems of cube duplication and angle trisection and the linear nature of the problem of the generalized angle division, oblivious, it would appear, even to the realization that such a claim required a proof. Indeed, Pappus prefaces Book III of the *Collection* with scornful remarks on the attempt by a colleague to solve the cube duplication via planar methods.⁴³ To be sure, the method is unsuccessful, being at best a procedure for the arbitrarily close approximation of the solution via a recursive planar construction. But Pappus speaks almost as if the very notion of trying to find such a solution for the "solid" problem is absurd *per se*. Like the mathematical authorities of a much later age who effectively ruled against the possibility of the circle quadrature a century before the transcendentality of the problem was actually demonstrated,⁴⁴ he appears to have accepted the long record of failed attempts as the basis for affirming the solid character of the cube duplication:

Admitting the problem to be solid, they [e.g., Eratosthenes, Philo, and Hero] effected its construction only by means of instruments—in agreement with Apollonius of Perga who also effected its analysis by means of the sections of the cone, while others [did so] by means of the solid loci of Aristaeus, but no one by means of properly so-called planar [methods].⁴⁵

However convenient the classification might be, we would surely have appreciated from him some recognition that the formal placement of a solid problem like this is an open question subject to proof.

The situation with respect to loci is subtly different. A "planar locus" is by definition a straight line or a circle; a "solid locus" is one of the conic sections. If, then, the determination of a problem of locus results in one of the former, the locus is planar; if one of the latter, it is solid. There can be no ambiguity here, so that the classification of loci and problems of locus into planar, solid, and linear categories is perfectly feasible. As indicated above, the terminology

is already well attested with Apollonius and well suits the aim of systematizing the study of constructions in the analytic works by him and his contemporaries. What remains difficult to explain, however, is the rationale of the terms "planar" and "solid" in the instance of loci, unless we are to accept the commentators' very weak proposals, as considered above. It seems to me that an account of the following sort is possible. Well before Apollonius, one already knew that loci of the former sort, namely lines and circles, had their special domain of application toward the solution of problems in which a line segment must be found to satisfy a specified second-order relation with respect to given lines. Similarly, loci which are conics were known to be applicable toward the solution of problems in which the line segment must satisfy a third-order relation. This aspect of the applicability of these two types of loci could justify naming them "planar" and "solid," respectively, even if one did not have a clear proof that all problems expressible via such second- and third-order relations can be solved via loci of the former and latter types, respectively. The naming would surely be descriptive, rather than normative, and would be convenient for focusing research into the relative domains of problems solvable via the different methods of construction.

It is not difficult to see how one might go on to extrapolate this viable scheme for problems of locus to the whole field of geometric problems, in this way obtaining the classification used by Hero and Pappus. Although this move brings with it the logical difficulties just mentioned, the commentators seem unaware of this fact. It is thus possible that one of them bears responsibility for introducing the defective conception of the classification scheme, rather than its having been a fixture within the earlier research tradition. Since neither Proclus nor Eutocius follows this scheme, it would appear not to have received a conspicuous statement in the works of the earlier geometers or in the later compilations by Geminus and Menelaus. Far from being the "standard" ancient view, then, Pappus' tripartite division of problems and the criterion for the choice of construction techniques which accompanies it emerge as a minority opinion held by a few of the late commentators through their misconception of the nature of the 3rd-century enterprise of problem solving.

PROBLEMS, THEOREMS AND THE METHOD OF ANALYSIS

The distinction between problems and theorems is maintained meticulously by the authors of the principal treatises of classical geometry. Euclid, Archimedes, and Apollonius, as well as in the miscellaneous materials preserved by the later commentators, Hero, Pappus, and Eutocius. As a matter of *form*, a problem is cast as an infinitive expression seeking the construction of a geometric term in a specified relation to other given terms (e.g., "to construct the triangle whose sides are three given line segments").⁴⁵ By contrast, a theorem is typically set in the form of a conditional asserting a property of a specified geometric configuration (e.g., "if in any triangle two lines are drawn from the endpoints of

its base and meet within it, then the lines will have a combined length less than that of the two remaining sides").⁴⁷ A point not missed by the ancient commentators is that the theorem refers to a general class of entities (e.g., "any triangle"), whereas the problem usually results in the production of a unique figure. Indeed, it may happen that the problem cannot be solved unless certain auxiliary conditions are met. One thus typically supplies a "diorism" asserting those conditions (in the above case, "where the three lines are such that no two have a combined length less than the third"). This might be motivated through a separate theorem (as in Euclid, "in any triangle the combined length of any two of its sides is greater than its third side")⁴⁸ or alternatively, usually in the context of an analysis of the problem, through an allusion to a condition for a certain step of the construction (e.g., that two given circles must have a point of intersection not lying on the line joining their centers). When the problem has many solutions, it may be expressed as a theorem of locus (e.g., "the points whose distances from two given points have a given ratio lie on a given circle"), or as a locus problem (e.g., "to find the locus of points such that . . .").⁴⁹ Such loci provide an important instrument for the solution of problems. For instance, the problem of constructing the triangle of three given sides might alternatively be found via the intersection of a circle (the locus of points whose distances from two given points are in a given ratio) and an ellipse (the locus of points whose distances from the same two given points have a given sum). Of course, since this particular problem permits of solution within the context of elementary geometry, one could not here wish to appeal to the more advanced methods of conics, as this alternative construction does.⁵⁰

These illustrations indicate that from the purely formal viewpoint the distinction between problems and theorems is largely artificial. One can easily recast any problem as a theorem, merely by incorporating into the protasis of the theorem all the details of the construction of the problem. Conversely, a theorem might be rephrased as the demand to produce a construction or to discover some property of a given figure. In most instances, there would appear to be a natural preference for one form over the other. In the case of propositions of locus, however, the distinction is not entirely clear. Like problems, they require a construction; but like theorems, they aim to prove a property of what is constructed, namely, its identification as a given line, circle, conic, or other figure. Pappus and Proclus thus reserve a third category, "porisms," in separation from those of problems and theorems,⁵¹ and I suspect that this is nothing other than the class of loci. The commentators are neither clear nor convincing in explaining how porisms differ from problems, but they do assure us that Euclid's *Porisms* was a compilation of theorems of locus. I would thus suppose the "porism" was Euclid's term for what later geometers designated as "locus," and that, not perceiving this, the later commentators attempted their own explanation on etymological grounds.

A formal treatise in geometry might thus be organized as a body of theorems presented in the order of their deductive dependence. As Proclus observes, Euclid strikes a roughly even balance between theorems and problems in his first book.⁵²

In this way, the problems often serve as justification for the introduction of auxiliary terms in later propositions. For instance, the theorem that the angles of any triangle equal two right angles (I, 32) requires the introduction of a line parallel to a side of a given triangle and passing through one of its vertices; this can be effected through the construction set out in the preceding problem (I, 31). Commentators, both ancient and modern, have thus found it natural to assign to the Euclidean problems the special role of securing the proofs of existence of the geometrical entities investigated in the theorems.⁵³ We shall consider this issue further below. But one should note that this view offers, at best, an incomplete picture of the diverse role of problems within the ancient geometry. Some problems are included not for their applications in later theorems, but for their intrinsic interest. Book IV and most of Book XIII, for instance, are devoted to the exposition of problems: the inscription of regular polygons in the former, that of the regular polyhedra in the latter. Here, the subordination is reversed, in that any theorems enter as auxiliary lemmas toward the constructions.⁵⁴ Pappus' treatment of the angle trisection suggests a provenance in a deductively ordered structure of solved problems: the trisection assumes a *neusis* whose solution as a solid problem is known from a prior lemma; the latter, in turn, assumes the construction of a hyperbola of given asymptotes, solved in another lemma; this refers back to the construction of the conic sections answering to given parameters, as solved by Apollonius in Book I of the *Conics*.⁵⁵ Clearly, the structure of these problems is determined by the deductive requirements of the problems themselves, not by the demands arising from their use in other theorems. Indeed, Apollonius remarks that theorems like those expounded in *Conics* III have their principal value within proofs dealing with locus problems.⁵⁶ It is noteworthy that problems are few in number in the first four books of the *Conics*. As we have seen, the appearance of the hyperbola construction in II, 4 is due to an interpolation by or after Eutocius; the sections of problems at the end of Book I and closing Book II cannot easily be viewed as contributing to the deductive needs of the theorems in other parts of the *Conics*. Their removal would in no way affect the logical structure or soundness of the main body of theorems.⁵⁷ The same may be observed of Euclid's Book I. The validity of its theorems does not depend on the presence of the problems.⁵⁸ Were one to segregate the problems, however, the proofs of their constructions would require appeal to the theorems in the rest of the book. One may see that the three postulates of construction—to draw the line through two given points; to extend a given line indefinitely; and to draw the circle of given center and radius—are stated in the form of *problems* whose construction may be assumed as irreducible, and so may initiate the ordered sequence of problems. By contrast, the two remaining postulates—the equality of all right angles and the nonparallelism of two lines when the interior angles formed with them by a transversal line sum to less than two right angles—are stated in the form of *theorems* and are introduced as premisses in theorems in Book I.⁵⁹

Further examples of this kind could be supplied in profusion. They reveal that problem solving was the essential part of the geometric enterprise marked

off by the works of Euclid, Apollonius, and those in their tradition, and that for them the compilation of bodies of theorems was an effort ancillary to this activity. It is important to realize this as one turns to the views proposed by ancient thinkers on the nature of problems and theorems and their relative significance for geometry. Pappus and Proclus draw from a centuries-long tradition of discussion of the definitions of these and associated terms, of which the following, from Pappus, is a representative sample:

Those wishing to discriminate more precisely the things sought in geometry . . . think fit to call a "problem" that in whose case it is proposed (*proballain*) to make or construct something, but a "theorem" that in which, on the supposition of certain things, one investigates (*theōrein*) what is implied by them and is contingent on them in every way. Among the ancients, some call [them] all problems, others [call them all] theorems.⁶⁰

On this latter division of opinion, Pappus tells us no more; but Proclus reports the positions of half a dozen writers widely distributed over the span of antiquity. As one might anticipate, those emphasizing the role of theorems draw their inspiration from the ideas of Plato: since the objects of geometry are eternal, unchanging verities, it is improper to speak of one's bringing such things into being, constructing them, and so on, the way one typically does in a geometric problem.⁶¹ Rather, their study is the contemplation of what they truly are, and thus only the form of the theorem is appropriate to this discipline. Proclus ascribes this view to Plato's nephew Speusippus, as well as to a mathematician named Amphinomus.⁶² Much the same view is assigned to Plato himself by Plutarch, on the authority of the Platonic essay by Eratosthenes⁶³; even if the incident of Plato's chiding Eudoxus for his approach to solving the cube duplication is surely apocryphal, this conception is indeed faithful to Plato's own thought on the nature of geometry. As director of the later Academy and exponent of Neoplatonism, Proclus naturally finds his sympathies allied with the same view. In agreement with Geminus, he makes it the aim of geometry to move through problems to theorems, as it were, rising from sensibles to ideals.⁶⁴

By contrast, writers closer to the activity of geometers emphasize the contrary view. According to Proclus, the "mathematicians in the following of Menaechmus" classed all the propositions as "problems" (*problēmata*)⁶⁵:

while the problem (*problē*) is of two sorts. Sometimes the thing sought is produced (*porizein*); but sometimes by taking what has been delimited we see either what it is, or what kind of thing it is, or what property it has, or what relations it has to something else.

Here we perceive the same distinction which results in the separation of locus problems (and "porisms") from the body of problems of construction, although Menaechmus hardly subscribes to Euclid's terms. Nevertheless, it is an important insight that the methods of finding constructions might be applied toward the discovery of properties of given configurations. Proclus does not provide the rationale behind Menaechmus' view, although it would appear to be epistemological, in contrast with the ontological position of the Platonists. That is, the

"problem" is the superior vehicle in the heuristic effort of gaining knowledge about geometric figures. Proclus attempts a compromise by admitting that we must engage in an activity which employs a vocabulary of construction, but that geometry deals only with the "intelligible matter" of figures, rather than with their "sensible matter," as one does in mechanics.⁶⁶ The actual being of abstract geometrical entities, however, is subject neither to generation nor to change of any sort. Proclus, through his preoccupation with the Platonic doctrines on the nature of being, seems to have missed the key issue of heuristic method.

Another mathematical writer, Carpus of Antioch "the mechanician," defends the priority of problems on pedagogical grounds. Elsewhere he is known to us for an account of the mechanical work of Archimedes and for having solved the circle quadrature via "the curve called simply 'of double motion'."⁶⁷ To Carpus, problems have an obvious advantage over theorems in their being simple to state. Proclus cites, at some length, certain views drawn from his work "on the subject matter of astronomy":

on the distinction of these, he says that the problematic genus is prior to that of the theorems. For the subjects about which properties are sought are found through the problems. Moreover, the statement (*protasis*) of the problem is simple and without further need of any specialized understanding. . . . But that of the theorem is effortful and demanding of great precision and expert discrimination, in order that it appear neither more nor less than the truth.⁶⁸

Carpus illustrates this point in the particular context of the problems and theorems which open Euclid's Book I, and so has in mind the pragmatic question of how best to present geometric materials to learners. Proclus concedes the greater effectiveness of the problematic form in this case, as also in the case of the application of geometry to the mechanical arts. But he musters support from Geminus for the view of the greater perfection of theorems, on the grounds, as noted above, that these better capture the essence of abstract geometric truth. Proclus thus seems not to sense the significance of another statement by Carpus, that the usual methods for problems are well suited for "the more obscure cases." From the viewpoint of geometric research, this is surely the striking advantage of the problematic form: that one may, with relative ease, articulate a problem whose solution is not known and in this way define a direction for gaining new knowledge in geometry. By contrast, the statement of a theorem must already incorporate the full description of what has been discovered. The theorem, then, is the appropriate mode when one is concerned with the formal organization of known results, rather than with the discovery of what is not yet known. In their respective positions on the evaluation of the problematic and theoretic forms, Proclus and Carpus thus signal the contrasting objectives appropriate to their differing manners of involvement in geometric studies.

Proclus calls attention to another distinction between problems and theorems⁶⁹:

According to Zenodotus⁷⁰. . . the theorem is distinguished from the problem in that the theorem seeks what is the property predicated of its subject matter, while the problem seeks what is to exist (in order that) something be. Whence also, ar-

According to Posidonius,⁷¹ the one is defined as a proposition according to which one seeks whether it is or is not [possible], the other⁷² a proposition in which one seeks what it is or what sort of thing it is. . . . For it is different either to seek simply and indefinitely whether it is [possible to construct a line] at right angles from this point to this line, or to investigate [*theôrein*] what is [the nature of] the line at right angles.

Despite the highly elliptical terms, it is clear that both views assign to problems an existential function. In the former case, they effect the configuration such that a specified condition may be said to obtain. In the latter, they determine whether or not a construction satisfying the specified condition is possible. They thus accord quite comfortably with an account in the manner of modern logic, which would transcribe theorems via universal quantifiers, but problems via existential quantifiers. Indeed, Proclus subscribes to this view of problems in his own reconstruction of Euclid's dialectical purposes in first setting out three problems of construction (I, 1-3) before taking up the first theorem on congruence of triangles (I, 4):

Without a prior construction of the triangles and a production of their genesis, how could he presume to teach about their essential properties and of the equality of their angles and their sides? . . . For let someone, happening along before the making of those things, say that if two triangles have a property like that, they shall also have this one in every case. Then, would it not be easy for everyone to respond to him: For do we at all know whether a triangle is capable of being constructed? . . . To cut off these difficulties in advance, then, the author of the *Elements* has also handed down the construction of the triangles. . . and to these he quite reasonably attaches the theorem by which is proved [the property of congruent triangles asserted here].⁷³

However nicely this account suits the needs raised by Proclus' Platonist ontology, it hardly presents a convincing portrait of Euclid's intent. If he had indeed wished to affirm the existence of triangles, he need only have taken three arbitrary points and then introduced the three line segments which connect them two by two in accordance with his first postulate. The assumption of arbitrary points or lines as "given" is not founded on any explicit postulate, but is in fact an element in practically all of Euclid's problems. For instance, in I, 1, the line segment which is to serve as the base of the constructed equilateral triangle is such a given term. What is especially inappropriate about this construction as an alleged proof of the existence of triangles is that it brings about only one particular kind of triangle and might thus risk misleading the learner into conceiving of this special case, when in the context of general theorems on triangles, such as I, 4, he ought to conceive a case better representative of the general class. But as a simple yet interesting first exercise in the mobilization of postulates and other general principles for working out a formal geometric proposition, the problem Euclid has chosen to open Book I is beyond reproach.⁷⁴

It is of course an important insight on the part of Proclus and the commentators on whom he depends to perceive this nuance in the logical conception of problems, and we would certainly not wish to deny that Euclid and other geometers

might themselves have been aware of it. But as we have seen in the many examples referred to earlier, this view cannot cover the full range of contexts served by the introduction of problems in the ancient geometry. Indeed, this existential aspect is strongly marked only in the case of "diorisms," which articulate the limits on the solvability of problems. Note, however, that the condition of solvability for the problem of constructing a triangle of given sides (I, 22) is framed by Euclid as a *theorem* (I, 20): that any two sides of a triangle have a combined length greater than the third side; not as a *problem*.⁷⁵ Euclid does not actually show how this result secures the existence of the point of intersection of the two circles by which the construction of the triangle is effected in the subsequent problem. It has thus been left to Proclus and other commentators to fill this gap.⁷⁶ In view of Euclid's omission, one becomes doubly wary of the suppositions that he appreciated the need for existence proofs in cases like this, and that he inserted the problems to serve this need.⁷⁷

Throughout the present study, we have seen the heuristic power which the method of analysis afforded the ancients in their search for solutions to geometric problems. The most detailed account of this method surviving from antiquity is given by Pappus as an introduction to his survey of the analytic corpus in *Collection* VII. To this we may add two very brief statements on the nature of this method, the one in a scholium prefacing some alternative proofs to the opening theorems of Euclid's Book XIII, the other from Hero's commentary to Euclid's Book II as extant in the Arabic translation by al-Nairizi.⁷⁸ A comparison of the three versions reveals some important aspects of Pappus' account which seem not to have been considered in the profuse discussion of it by modern scholars.⁷⁹

Pappus opens with a statement summarizing the nature and objectives of the analytic corpus (*topos analyomenos*):⁸⁰

The so-called 'analytic [corpus]' . . . is a certain special body prepared next after the making of the common elements for those wishing to acquire a power in lines [i.e., lines and curves] conducive to the finding of problems proposed to them, and it has been compiled as useful for this purpose alone. It has been written by three men: Euclid, author of the *Elements*, and Apollonius of Perga, and Aristaetus the Elder, and it adopts the method according to analysis and synthesis.

He now sets out to characterize this special method for problem-solving:

(Pappus) Now analysis is a way from what is sought, as if admitted, through the things that follow in order, up to something admitted in the synthesis.⁸¹

(Scholium) Now analysis is a taking of what is sought, as if admitted, through the things that follow, up to something admitted as true.⁸²

(Hero) Now analysis is, when any question has been proposed, that we first set it in the order of the thing sought, which has been found [!], and we will then reduce (it to something) whose proof has already preceded. Then therefore it is manifest, we say, that the thing sought has now been found according to analysis.⁸³

The statements by Pappus and the scholiast are obviously in virtual literal agree-

ment. An actual dependence of the scholiast on Pappus is entirely possible; but the conformity in sense (if not in specific wording) with Hero reveals that at least this much of the account has a firm precedent in sources older than Pappus. From here, Pappus amplifies thus:

(Pappus) For in the analysis, having hypothesized what is sought as if already in effect, we examine that from which this results and in turn the antecedent of that, until proceeding in this backward manner we come down opposite some one of the things already known or having the order of first-principle. And such a method we call 'analysis,' as being like a backward resolution.

Through our experience of the ancient uses of analysis, we should have expected it to be described as a searching through the geometrical consequences deduced from what is sought, on the hypothesis that the latter is already in effect. This is the natural reading of the opening lines, even if the term "things that follow" (*ta akoloutha*) need not bear the stronger sense of logical deduction in a context like this one.⁸⁴ Pappus, however, seems to view the procedure in quite a different manner, as a finding of successive antecedents, so that the result initially hypothesized ends up being a deduction from the derived sequence. But before pursuing this further, let us note that Pappus' remarks have the appearance of being a gloss on a statement on the analytic method received from a prior source. A similar pattern emerges in the next section of his account. The two alternative accounts provide the anticipated analogues to their statements of analysis:

(Scholium) But synthesis is a taking of the thing admitted through the things that follow up to something admitted as true.⁸⁵

(Hero) But synthesis is, that we begin from a thing known, then we will compose, until the thing sought is found. Thus, the thing sought will then be manifest according to synthesis.

Pappus omits any such corresponding line, but instead provides the complement to his own amplified statement on analysis:

(Pappus) But in the synthesis, in reverse order, having posited as already in effect what had been obtained last in the analysis, and having ordered in the natural manner as consequents what there [were ordered as] antecedents,⁸⁶ and having composed them to each other, we at last arrive at the construction of what is sought. And this we call 'synthesis.'

For Pappus, then, the difference between the two methods is merely one of the order of exposition. Despite his earlier insistence on the singular significance of analysis for the study of problems, he now accords with Hero and the scholiast in considering its application in the instance of theorems. To establish a theorem analytically, then, Pappus would hypothesize as true the desired conclusion A and then reason, "A will follow, if B holds; and B, if C; etc." Arriving at something known to be true (e.g., a previous theorem or a postulate), he can then affirm the truth of A. Now, Hero and the scholiast go on actually to work out demonstrations in the double mode for a series of theorems.⁸⁷ In the former instance, this results in alternative proofs for eight of the first 10 theorems in

Euclid's Book II; in the latter, for the five theorems which open his Book XIII. The theorems are all quite straightforward, and both writers treat them in the same manner. Far from adopting the mode suggested by Pappus, both frame the analysis as a sequence of deductions leading from the hypothesis of the result to be proved and terminating in something known to be true. In effect, their analyses are nothing other than regular proofs of the converse of the proposed theorems. Furthermore, one may note that both writers share an interest in securing their theorems without recourse to the geometric diagrams, this being possible here in that only the formal manipulation of identities is at issue. Such affinities between these two treatments must surely indicate mutual dependence, either through common authorship or through the scholiast's referring to Hero for his model. In the light of our earlier observation of the agreement between the scholium and the opening lines of Pappus' account, this would suggest that it is Pappus who has borrowed from a source (for instance, Hero himself or a writer following him) to establish a familiar context for the elucidation of this method. The remarks which he attaches to it, however, would reflect his own interpretation of the ancient geometry, but have no particular standing within the older tradition of mathematical commentary.

A fundamental difficulty in Pappus' view of the order of deductive sequence in the analysis is that, if he were correct, there ought to be no need for the synthesis; for the analysis of itself would constitute a demonstration of the theorem. It happens that a treatment adopting precisely the form implied by Pappus appears among his lemmas to Apollonius' *Determinate Section*.⁹⁰ In the context of a specified figure, it is to be proved that a stated relation A obtains; the writer passes through a series of steps of the form "B is the same as C" and "if D obtains, it will be necessary alternately to seek whether E obtains." He ultimately hits upon a relation known to obtain. In his paraphrase of this lemma, Heath speaks of it as "a case of *theoretical analysis* followed by *synthesis*," and after the analysis he adds the words, "The synthesis is obvious." To be sure, this tag line appears now and again among the analyses in the *Collection*.⁹¹ But in the present case it is purely Heath's addition; for it is not used here by Pappus, and there is nothing which overtly indicates that this proof is intended as an analysis or that a synthesis ought to be supplied. Thus, if the writer of the lemma is following the procedure prescribed by Pappus, replacing each derived relation with another equivalent to it, he treats the synthesis not merely as "obvious," but as superfluous.⁹²

One can see how Pappus might construe a proof technique of this sort to be a form of analysis. Reflecting on passages in the philosophical literature where the analytic manner is pictured as a search for antecedent premises or conditions, he might then adopt this as his own view, as he sets out to amplify the sketchy remarks made by earlier mathematical commentators. The following passage from Aristotle's discussion of the nature of moral deliberation (*bouleusis*) is especially interesting in this connection⁹³:

The statesman does not deliberate over good order, nor does anyone else over the end [to be achieved]. But *positing the end*, they *investigate* the questions of how

and through what things it shall be. If it appears to result from several things, they *investigate* through what it will result in the easiest and best manner; but if it seems to result from a single thing, they consider how it shall result from that, and what that in turn results from, until they *arrive* at the first cause which is last in the order of discovery. For the deliberator seems to *seek* and to *analyze* in the said manner, just as in the case of a diagram [i.e., geometric figure or proposition] . . . , and that what is last in the *analysis* is first in the genesis. Now if they *happen upon* something *impossible*, they leave off; for instance, if it requires money, but this cannot be *provided*. But if it appears *possible*, they try to do it.

Forms of the terms indicated in italics appear in Pappus' account of analysis as well.⁹⁴ It fixes the method as a search for antecedent causes in a way that examples drawn from the mathematical literature never would. Even Aristotle's consideration of the two alternative outcomes of the analysis figures prominently in Pappus' subsequent remarks. We are thus readily led to suppose that Pappus drew the basis of his account of analysis from the philosophical literature. Of course, Pappus does not cite Aristotle here.⁹⁵ But nowhere in the *Collection* does he cite any philosopher other than Plato, and that only once for a philosophical view.⁹⁶ The several references to the "five Platonic solids," that is, the regular polyhedra (in Pappus' Book V), can hardly be viewed as citations of Plato. One can discern several Aristotelian doctrines in Pappus' *Commentary on Euclid's Book X*, even though he does not expressly cite Aristotle there either.⁹⁷ It seems to me most probable that Pappus could pick up such general views through the medium of commentators like Geminus and others, conversant with a syncretizing form of Platonism.⁹⁸ In such a case, Pappus himself might not be fully aware of the ultimate provenance of some of his views.

One thus approaches the concluding section of Pappus' account with suspicions that it, too, despite his access to the whole ancient corpus of analysis, may present not a distillation of that ancient tradition, but rather a rephrasing of standard philosophical views. He is concerned here with distinguishing between the analysis of theorems and that of problems:⁹⁹

The class of analysis is of two sorts: the one which is called 'theoretic' seeks what is true, while the other which is called 'problematic' is productive (*poristikos*) of what has been proposed. In the case of the theoretic kind, having hypothesized what is sought as being and as true, and then having gone forward through the things which follow in order, as true and as being in accordance with hypothesis, up to something admitted, then if that thing admitted is true, so also will the thing sought be true, and the proof will be converse to the analysis; but if we come upon a thing admitted as false, so also will the thing sought be false.

In the case of the problematic kind, having hypothesized the thing proposed as known, and then having gone forward through the things which follow in order as true up to something admitted, then if that admitted thing is possible and producible, what those in mathematics call 'given' (*dothen*), so also will the thing proposed be possible, and in turn the proof will be converse to the analysis; but if we happen upon something admitted as impossible, so also will the problem be impossible.¹⁰⁰

And a 'diorism' is a preliminary distinction of when and how and in how many ways the problem also will be possible.

So much then concerning analysis and synthesis.

This amounts to hardly more than an elaboration of the preceding remarks in the obvious way. Pappus remains uncertain about the order of implication in the analyses, and in the strict sense he would require all steps to be simply convertible. Otherwise, he could not maintain both that the truth of what is derived implies the truth of the hypothesis and also that its falsity implies the falsity of the hypothesis. As we have seen, the effort of logically converting the steps in the analysis of a problem to obtain its synthesis is the basis for discovering the conditions under which the solution is actually possible, and so leads to the articulation of the *diorism*. But what Pappus has to say about this critical element of the method is so extraneous to his account that the modern editors typically bracket this line as a later interpolation.⁹⁹ Similarly, the extremely subtle role of the "givens" in the analysis, as we have observed in Chapter 4, is barely hinted at in Pappus' feeble aside. Applying insights derived from the modern logical investigation of heuristics, Hintikka and Remes have perceived nuances in the logical status of auxiliary constructions in the analysis of figures.¹⁰⁰ From Pappus' account one may gather, however, that the commentator was at best but subliminally aware of such difficulties.

What Pappus emphasizes is the distinction between the theoretic and problematic forms of analysis. But examined more closely, one begins to see that his notion of an analysis of theorems is gratuitous. To the extent that he is viewing the expository form of the method, rather than its heuristic role, he seems not to sense that such analyses are entirely redundant¹⁰¹; for consider any one of the cases worked out by Hero or the scholiast to *Elements* XIII, for instance, the analysis and synthesis of the theorem XIII, 1, to the effect that if a line is divided into segments x and y in extreme and mean ratio (i.e., $y : x = x : x + y$), then the square of $\frac{1}{2}(x + y) + x$ is five times the square of $\frac{1}{2}(x + y)$. In the analysis one adopts the hypothesis that $(z + x)^2 = 5z^2$ and then deduces through the condition that x and y are in extreme and mean ratio that $z = \frac{1}{2}(x + y)$; in the synthesis one starts from $z = \frac{1}{2}(x + y)$ and deduces that $(z + x)^2 = 5z^2$. Clearly, the presentation of the analysis here is completely artificial. In the case of the more complicated geometric configuration given by Pappus in a lemma toward the construction of a case of Apollonius' three-circle problem, the steps leading to the desired proof might be less obvious, so that the artificiality of the analysis is less evident.¹⁰² But what the commentators lose sight of is that such theorems arise within the context of other theorems or problems which not only reveal what the claim made by the theorem ought to be—that is not merely a matter of conjecture—but also will display the elements of the desired proof. It is no wonder, then, that analyses of theorems are so rarely found in the mathematical literature. These are either artificial and unnecessary (as in those given by Hero) or serve for finding proofs to theorems whose stated result is somehow known in isolation from the context which gives

rise to them. Note, furthermore, that in Pappus' alternative outcome of the analysis of the theorem, where the derived result turns out to be false, the formal demonstration may be produced merely by negating the hypothesis to obtain a new assertion and then submitting the steps of the analysis, as already worked out, in proof of this. The term "reduction to the impossible" designating this form of proof was already familiar at the time of Aristotle, and doubtless even before then.¹⁰³ By its very name, it invites comparison to the method of "reduction" used, for instance, by Hippocrates in the examination of the cube duplication, and discernible as a precursor of the method of analysis. The connection was not missed by the later commentators.¹⁰⁴ What this reveals is that for a large body of theorems, the analysis of itself yields a proof, so that no synthesis will be called for. Once again, the artificiality of Pappus' theoretic type of analysis and synthesis is apparent.

It is not difficult to explain how Pappus came to advance this notion of theoretic analysis.¹⁰⁵ If we again consider the theorem of the scholiast, the result it establishes is needed in XIII, 11 for showing that if a regular pentagon is inscribed in a circle of rational diameter, its side will be a "lesser" irrational line.¹⁰⁶ The construction here gives rise to the division of a given line into extreme and mean ratio and the consideration of the ratio of $(z + x)^2$ to z^2 , for $z = \frac{1}{2}(x + y)$. One might well adopt the format of an analysis to determine what that ratio is, namely 5 : 1. But in doing so, one would be analyzing not a theorem, but a *problem*.¹⁰⁷ We recall that Menaechmus had indicated two types of problems: those that lead to the construction of a figure having certain specified properties, and those that lead to the determination of certain properties obtaining for a specified figure. In the latter case, the result would naturally be stated as a theorem, of which the proof would be identical to the synthesis of what was earlier proposed as a problem. But having made this determination, one would have neither expository nor heuristic reasons for fabricating an analysis in addition to the synthesis of this derived theorem.

In this regard, special interest attaches to an observation made by Carpus in his advocacy of the superiority of problems over theorems¹⁰⁸:

And in the case of problems there is a certain single way, that found in common through the analysis, according to which by going forward we can achieve success. For in this way one hunts after the more obscure of the problems. But in the case of theorems the handling is hard to get hold of, as up to our times, he says, no one has been able to offer a common method for the discovery of these, so that by virtue also of their facility the class of problems would be simpler.

Although Carpus does not insist that analysis is a method appropriate to problems to the exclusion of theorems, that is the natural inference to be drawn from his remark. He is known to antedate Pappus, who cites him, and we would surely set him before Hero as well. For the effort by Hero and Pappus to accommodate the method of analysis to theorems, indeed to suggest in subtle ways that its use for theorems might even be prior to that for problems, would surely have pressed Carpus not to deny the *existence* of a proposal for a general method of inves-

tigating theorems, but rather to question the alleged *utility* of analysis in these cases. Viewed thus in the light of Carpus' position, Pappus' account of analysis gains a dialectical dimension. By insisting on the relevance of this method to the study of theorems, he effectively neutralizes one of the more telling arguments on the side of the problems in the continuing debate over the relative importance of problems and theorems.¹⁰⁹ But he does this without convincing support from the mathematical literature, where the instances of theoretic analysis, rare on any account, all seem to be due to later commentators, like Pappus himself, as they supply minor lemmas to the great treatises of the older geometric tradition.

Blinded by the glare of Plato's ontology, the later commentators seek to impose on the classical geometry an interpretation it does not easily sustain. Since problems seek to construct geometrical entities, that is, to bring into being things which are forever existing, or seek to cut, augment, diminish, or otherwise modify things which are in reality immutable, they must assume a position subordinate to theorems. For only the latter are philosophically correct in form, proving what is true of the ideal entities of geometry. Problems are allowed a role at the tentative stage when knowledge is incomplete, or for the convenience of learners. Or they might render the service of effecting proofs of the existence of terms needed in the formal demonstrations of theorems. Even the method of analysis, the method *par excellence* of problem solving, is transferred to theorems through the artificial ploy of "theoretic analysis," while in their quest for formal proofs the commentators regularly choose to delete the analyses from older treatments cast in accordance with the double method.¹¹⁰ In all these ways the later tradition betrays its insensitivity to the outlook and objectives of the older enterprise of geometry. To be sure, there are rare occasions when the commentators somehow grasp the discrepancy. In his discussion of Euclid's *Porisms*, for instance, Pappus observes¹¹¹:

... the ancients best knew the difference between these three kinds, as is clear from the definitions; for they said that a "theorem" is what is put forward for the proof of the thing proposed, a "problem" is what is projected for its construction, and a "porism" is what is put forward for its production (*porismos*). But this definition has been changed by the more recent [writers] who can't produce (*porizein*) everything, but use these elements and prove only this, that what is sought exists,¹¹² but don't produce this.

This statement must discourage any supposition that the ancients conceived of problems merely as existence lemmas for the proofs of theorems. In so distancing himself from those contemporaries who supposed this very thing, Pappus for once displays a feeling for the heuristic motives of the older generation of geometers. For Apollonius and the other authors of the analytic corpus, the discovery of solutions to problems was the heart of their geometric activity, and by their own account, the compilation of theorems was subordinate to this. But for the later generations, not sharing this heuristic motive, the power of the analytic method became merely superfluous—impressive, but no longer valuable.

"... AND MANY AND THE GREATEST SOUGHT, BUT DID NOT FIND."

The search for solutions to the three problems of cube duplication, angle trisection, and circle quadrature spanned the entire period of the classical geometry from Hippocrates to Apollonius and retained great interest for commentators throughout the later period of antiquity. Did the ancients view this effort as having succeeded in attaining its goal?

In the cases of the cube duplication and the angle trisection, the commentators seem pleased that the solutions put forward answer to the nature of the problems. Hero and Pappus designate them as "solid" problems, in view of the successful application of conic sections toward their solution. Of course, this need not of itself rule out the possibility of finding a more elementary "planar" method. But Pappus' almost abusive treatment of a certain colleague "pretending to be a great geometer" for his alleged solution of the cube duplication by planar means indicates that, to him, the book on this question would appear to have been closed.¹¹³ By analogy, we might expect that the several known constructions of the circle quadrature would leave him content to class the problem as "linear" and so to cease any search for a more elementary method. Indeed, he does just this in the case of the arbitrary division of the angle, classifying it as "linear" without a word on the possibility or advisability of looking for solutions other than the ones via the quadratrix and the spiral which he presents.¹¹⁴ But he never actually makes this move for the circle quadrature, so that he may well have wished to keep the question open. At least one of the ancient solutions, the fallacious attempt by Bryson to equate the circle with a rectilinear figure somehow intermediate between rectilinear figures (squares?) drawn around and within the circle, suggests the aim to find a solution via the familiar elementary methods.¹¹⁵ Is this indeed the geometers' objective in their research on this problem? The theme of the circle quadrature surfaces on several occasions in the discussions of the Aristotelian commentators, thanks to a few passing references to the problem in Aristotle's *Categories* (Ch. 7), *Posterior Analytics* (I, Chs. 9 and 12), and *Physics* (I, Ch. 2). Despite the nontechnical character of these sources, the commentators often avail themselves of authorities from the technical literature as well as from the historical and philosophical writings on geometry. Thus, one might expect that they would reflect whatever conclusions had been drawn concerning the status of the circle quadrature among the ancient geometers. In this indirect way, then, they are potential witnesses to the progress of these studies.

Philoponus, as we have noted, makes the important distinction between the *existence* of the square equal to the circle and its actual construction; geometers *assume* the former, he maintains, as the basis of their investigation of the latter.¹¹⁶ A similar observation is made by Eutocius in connection with Archimedes' theorem on the area of the circle. In proving that the circle equals the right triangle whose legs equal, respectively, the radius and the circumference of the

circle, has he not assumed that one can take a line equal to the circumference, a thing which has been proved neither by him nor by anyone else?

But one must immediately observe that Archimedes writes nothing outside of the things which are appropriate. For it is somehow clear to everyone that the circumference of the circle is some magnitude, I believe, and this is among those extended in one [sc. dimension], while the straight line is of the same kind. Even if it seemed not yet possible to produce a straight line equal to the circumference of the circle, nevertheless, the fact that there exists some straight line by nature equal to it is deemed by no one to be a matter for investigation.¹¹⁷

This is explicit admission of an appeal to intuitions of continuous magnitude, like several we have already cited, where the actual construction of the entity required is sharply separated from the issue of its existence.¹¹⁸ Speaking as a mathematical commentator, Eutocius is clearly of the view that no satisfactory construction of the solution to the circle quadrature has yet been achieved. Such is also the view of the philosophical commentator Ammonius, disciple of Proclus and mentor of Eutocius and Philoponus. In his remarks on Aristotle's distinction between potential and actual knowledge (*Categories*, Ch. 7), he observes:

The geometers, on constructing the square equal to the given rectilinear figure, sought whether it was possible to find a square equal to the given circle. And many and the greatest [of them] sought, but did not find it. Only the divine Archimedes found an extremely good approximation, but the exact construction has not been found to this day. And this is perhaps impossible; for on these grounds even he [sc. Aristotle] has said "if indeed it is knowable (*epistêton*)."—And perhaps for this reason he has made the straight line not dissimilar to the circular arc, if it is knowable or if it is not.—Then he says that [if indeed] the quadrature of the circle is knowable, although no manner of its knowledge yet exists, then also from this what is knowable is prior to knowledge.¹¹⁹

The remark is reproduced with further elaborations by Philoponus and by Simplicius.¹²⁰ From this it would appear that the commentators leave the issue open: the search for a construction is valid, that is, it is in principle possible, even if no successful result has yet been achieved. In maintaining the latter, Ammonius might seem merely to be ignorant of the more advanced constructing efforts, for he mentions only Archimedes' derivation of the estimate ($3 \frac{1}{4}$) in the *Dimension of the Circle*. But the same view, as we have seen, is held by Eutocius, who can hardly be charged with such ignorance. Simplicius too is of this opinion, while at the same time knows through a report derived from Iamblichus of the constructions via special curves proposed by Archimedes, Nicomedes, Apollonius, and Carpus.¹²¹

The same distinction between the known and the knowable appears in the commentary on Euclid's *Data* by Marinus, a disciple of Proclus, active late in the 5th century A.D. In specifying the variety of senses in which terms may be spoken of as "given," Marinus introduces the class of "produceables" (*porima*), namely, those entities whose construction can in fact be presented.¹²² Contrary to these are the "nonproduceables" (*apora*), of which the circle quadrature is

an example. But Marinus goes on to distinguish two sorts of entities in this latter class: those not yet constructed, but capable of construction (these are termed *poriston idiōs*, "produceable in the special sense," as contrasted with those in the former class, termed *kyriōs porimon*, "produced in the strict sense"); and the *apora*, opposed to the *porima*, whose constructibility (*zētêsis*) has not yet been decided (*adiakritos*). Alternatively, he terms the circle quadrature "unknown" (*agnōston*) yet "ordered" (or "fixed," as it were of constant value—*tetagmenon*). However pedantic this straining over terminological niceties might seem, the distinction Marinus is here striving to make between what is unknown and what is unknowable is an important one. Despite the long history of inconclusive efforts on the circle quadrature, the ancient mathematical tradition remained unwilling to pronounce on the impossibility in principle of finding a solution.

Closely related to this is the question of whether the diameter and the circumference of the circle are commensurable with each other. The development of geometric and arithmetic techniques for deciding on the rationality or irrationality of given terms, such as the incommensurability of the side and diameter of the square, was an important factor underlying the rigorizing movement within the pre-Euclidean geometry.¹²³ Nevertheless, Greek geometers might sometimes assert a conclusion of this kind without the firm support of a demonstration. Consider Hero's treatment of the problem of cutting off a segment equal to one-third a given circle (*Metrika* III, 18).¹²⁴ He explains that he will give only a convenient approximation to the solution, "since it is clear that the problem is not rational." That his solution is not exact is plain; for by finding two chords drawn from a point on the circumference, he shows that the area lying between the chords is exactly one-third of the circle, so that a small additional segment must be neglected in viewing this to be a solution to the stated problem. But on what grounds Hero can conclude the irrationality of the problem we learn nothing more. In the matter of the irrationality of π , comparable statements are made by medieval and Renaissance writers like Maimonides, Maurolico, Stifel, and Gregory, all long before the first actual proof of this result was obtained by Lambert in the 18th century.¹²⁵ Might these conjectures have been based on ancient precedents? Another remark from Eutocius' commentary on the *Dimension of the Circle* is suggestive. In defending Archimedes against the charge raised by some critics that his values for the ratio were only rough approximations, as compared with the closer values worked out by Apollonius and Ptolemy, Eutocius writes:

But if anyone felt strongly about effecting this in minute detail, he could have used what is said in the *Mathematical Syntaxis* of Claudius Ptolemy. . . and I myself would have done this, save that I had in mind, as I have said many times, that it is not possible exactly to find the straight line equal to the circumference of the circle by the things said here.¹²⁶

Given that Eutocius earlier observes, quite correctly, that the square root of a nonsquare number "cannot be found exactly,"¹²⁷ his reader might well be led

to infer that Eutocius likewise maintained the irrationality of the circumference and diameter of the circle, even if he does not flatly say that. Eutocius himself probably intends something weaker: that by qualifying his statement with the phrase "by the things said here," he means only to claim that *the specific procedure* set out by Archimedes cannot yield an exact value, since among other things it requires a series of root approximations. Doubtless, within the long tradition of practical computation the constant recourse to approximations for the measurement of circles and similar figures would occasion surmises concerning their irrationality. But the ancient geometers did indeed make the essential distinction between experience and proof in this instance, as is revealed in this comment by Eutocius' contemporary, Simplicius:

The reason why one still investigates the quadrature of the circle and the question as to whether there is a line equal to the circumference, despite their having remained entirely unsolved up to now, is the fact that no one has found out that these are impossible either, in contrast with the incommensurability of the diameter and the side (of the square).¹²⁴

In view of this firm statement recognizing the need for a demonstration, not yet at hand, of the impossibility of solving these problems, we may infer that Simplicius maintained the same caution relative to the question of their irrationality. On these matters the ancient tradition thus admitted that the goal of their search remained elusive.

What we never learn from the ancient writers on the circle quadrature, however, from the time of Aristotle and Eudemus in the 4th century B.C. right through to the time of the commentators in the 5th and 6th centuries A.D., is what the precise constraints on a construction would be for any solution of this problem to be judged acceptable. But certain responses to the constructions which were proposed offer us some insights. Speaking of the methods used by Archimedes and others, Simplicius remarks that "it would appear that they all made the construction of the theorem mechanical (*organikē*)".¹²⁹ In the same context, but now responding to a claim made by Porphyry (that "most learned" neo-Platonist commentator of the 3rd century A.D.), he remarks that "perhaps a certain mechanical discovery of the theorem had been made, but not a demonstrative (*apodeiktikē*) one".¹³⁰ Simplicius appears to maintain that a merely mechanical construction, corresponding to the use of curves of the "linear" class, does not satisfy the conditions for a geometric demonstration. In this, he conforms to an attitude already expressed by Pappus in his reservations over the "rather mechanical manner of generation" of the quadratrix, for which he can find a precedent in the criticisms made earlier by Sporus.¹³¹ A similar view arises among the Arabic geometers, who distinguish between the "mobile" and the "fixed" sorts of geometry, doubtless by way of transferring the ancients' distinction between "mechanical" and "demonstrative," and they too manifest a certain preference for the latter.¹³² We have already seen how Plutarch, a thinker with strong Platonist leanings, cast Plato himself in the role of criticizing the supposedly mechanical methods of Eudoxus and Menaechnus for solving the

cube duplication, and he even presumes to impute to Archimedes an antimetachanical outlook on the field of geometry.¹³³ But the later commentators are hardly unanimous in adopting this biased attitude. Geminus and Carpus acknowledge the great importance of mechanical pursuits in the mathematical work of Archimedes, even if his writings stress the precision of geometric theory.¹³⁴ In offering a mechanical method of his own for solving the cube duplication, Pappus shares in the spirit of the geometers of the 3rd century B.C., like Eratosthenes, Nicomedes and Diocles, whose search for practical solutions to such problems led them to explore mechanical approaches.¹³⁵ Both Hero and Pappus wrote extensively on the application of geometric techniques in mechanics. It is such a context which occasions Hero's presentation of the *neusis* for the cube duplication, for instance, and one readily detects the practical interest attached to his accounts of the measurement and section of figures in the *Metrika*.¹³⁶ When Pappus writes that "those problems in mechanics called 'organic' arise through being separated from the geometric domain," his meaning might not be entirely clear.¹³⁷ But it becomes so when we consider the item for which this serves as preface: the solution of the problem of gauging the diameter of a mutilated cylindrical column; for this leads, through a series of lemmas on ellipses, to the determination of the ellipse passing through five given points.¹³⁸ This is but one of many instances where the ancient geometers found in mechanical situations a source of interesting problems for geometric research.

When commentators like Simplicius voice their dissatisfaction with such mechanical approaches to the circle quadrature, they still do not clearly say what the alternative "demonstrative" or "geometric" construction ought to consist of. Presumably, their hope was for an elementary planar solution, or even a solid one. But if efforts along these lines actually were launched, evidence of them has not survived. To the extent that they insist on some such method, they also reveal an insensitivity to the origins of geometric constructions and means of construction; for how are the "planar" techniques of circles and straight lines any other than mechanical, that is, derived by way of abstraction from the practical use of compasses and rulers?¹³⁹ Similarly, the conics are no more or less mechanical than the curve formed by Archytas through the sectioning of solids. It is only through an arbitrary process of *fiat* by postulate that Euclid transforms these mechanical means into the starting points of a geometric theory. But for the conics and the field of solid geometry, even this process remains incomplete; for neither Euclid nor Apollonius eliminates the appeal to mechanical conceptions, specifically the rotations of plane figures about fixed axes, for the generation of cones and other solid figures, nor does either ever specify what the process of sectioning solids like these involves, whether by postulate or by theorem.¹⁴⁰ That the process is not merely conceptual is evident from the discussions of solid figures by Hero and Pappus, mentioned just above.

The situation relative to the constructing technique via *neusis* is the same. In some instances, as in the cube duplications of Philo, Hero, and Pappus, an actual ruler (*kanonion*) is conceived as being manipulated about a pivot

until the desired configuration is achieved.¹⁴¹ But in other instances, the operation is expressed in abstract terms: to draw a line such that, while inclining (*neuoussa*) toward a given point, it marks off segments having a specified property. Such is the mode adopted by Hippocrates (in the account from Eudemus) for the construction of the third lunule, by Archimedes in the lemmas to the theorems on spirals, and by Sporus in his cube duplication (an alternative equivalent to the manner followed by Diocles and Pappus).¹⁴² Such also is the manner of the cube duplication of Apollonius, as reported by Philoponus; it is nothing other than the abstract equivalent of the ruler construction given by Philo. Where the condition of the inclining line enters, one reads, "but this is assumed as a postulate (*aitēma*) without proof (*anapodeikton*)."¹⁴³ Now, this is surely a strategy one might have adopted: to posit a few of the generalized configurations of *neuses* as formal postulates and from this begin to elaborate a sequence of problems and theorems. But the ancients never seem to have taken this approach. For Apollonius and his colleagues, *neuses* were treated as problems to be effected in some instances, as for the cube duplication here or for the angle trisection, by means of conic sections, while in other instances by planar means.¹⁴⁴ Similarly, the motion-generated "linear" curves, like the spirals, conchoids and quadratrices, each have their own "primitive property" (*archaikon symptōma*); for instance, for the plane spiral it is that the rays drawn to the curve from the origin have the same ratio as the angles they make with the initial ray. All further properties and applications of these curves are effected as deductions from their *symptōmata*. If these curves are naturally conceived through mechanical motions, they need not have been treated as such in the formal exposition of their properties. That this formalization was not done, with the result that these curves came to occupy a subordinate position in the hierarchy of construction techniques, is thus largely accidental, the fault of their relative unfamiliarity, their lateness of invention, their more advanced character, or their specialized range of applicability, by comparison with the constructions presented by Euclid and Apollonius.

The central issue still eludes us: the sheer diversity of solutions proposed for each of the three special problems would indicate that the ancient geometers were engaged in a search never consummated to their satisfaction. What was the object of that search? The response, that they sought planar solutions for these problems, attempts to provide a simple motive behind the ancient geometry. It is a view which the commentators in late antiquity appear to maintain, as do many—indeed, most—modern scholars who write on this subject. But it is surely a brutal oversimplification; for in trying to probe the motives of researchers, we face an issue which has psychological aspects as well as formal mathematical ones, and this is likely to require a more complex account. Part of the answer, I believe, lies in recognizing a certain symbiosis between problems and the methods for solving them. The activity of studying and solving (or even not solving) problems gives rise to new techniques; the latter in their turn may gain a life of their own, and through further elaboration

give rise to new problems and new techniques. For instance, Archimedes' *neusis* for the angle trisection could suggest to Nicomedes the mechanical mode for generating the conchoids; but then, in addition to the initiating context of the problem, the curves draw attention to their asymptotic behavior and other properties and to the possibility of defining related curves, like the circle-based conchoids, and examining the applications of these. The later commentators do not sense this flexible interaction between problems and methods, this participation in a heuristic process, but instead attempt to describe it in terms of a specific product, the goal of research. But their inability actually to articulate this goal, to state precisely what form of construction was being sought within the earlier geometric tradition, reveals that no such specific goal operated as the motivating element for those geometers.

EPILOGUE

The Greek historical writer Herodotus tells the story of how Pharaoh Psammetichus set out to determine which of the races of man was the oldest.¹⁴⁵ He had two newborn babes carried off to be reared in isolation from society by a herdsman who was to report back as soon as they started uttering recognizable words. In the meantime he was under orders not to speak in their presence, but only to see to their physical needs, such as bringing in the goats every now and again to provide them milk. After about two years the herdsman noticed that the children would say *bekos* whenever he came to look in on them. He duly informed the king who found through further investigation that *bekos* was the Phrygian word for bread. For this reason, Herodotus says, the Egyptians from that day onward acknowledged the Phrygians as an older race than their own.

However much one might hope to discern in this account a prototype of the experimental method in the social sciences, it is clear that Herodotus is joking. The children were not speaking Phrygian, they were speaking goat. The ancient commentators on the analytic tradition of geometry slipped in much the same way. For the Aristotelian commentators, as also for the Neoplatonist Proclus, philosophical questions were naturally the principal interest. Geminus, significant for his encyclopedic survey of mathematics, was an adherent of the Stoic philosophy. Through him and his like, other writers such as Hero and Pappus, while engaged primarily in the presentation of mathematical texts, might thus import philosophical elements into their more general observations on the nature of mathematics. In this way, the later tradition of commentators applied the language and concepts of philosophy in their effort to describe, interpret, and criticize the works of the older tradition of geometers.

We have seen the fruits of their inquiry. Notions of the ideal nature of geometric entities lead them to emphasize the priority of theorems over problems for the exposition of geometric truths. The method of analysis, an effective instrument for discovering the solutions of problems, is purported to

serve as well in the investigation of theorems, a claim little encouraged through a consideration of the actual literature of geometry. Others, interested in giving an account of the logical structure of geometry, assign to problems the role of justifying the assumptions on the existence of entities made in theorems. Stimulating as an insight into the logical account of geometry, this view captures neither the diversity and power of the problematic form, nor the methods actually adopted by geometers for addressing issues of existence when they arose in their theorems. What the early geometers introduced as a useful division of loci becomes, among the commentators, a normative division of problems in general, imposing in an unworkable form the recommendation to seek planar methods of solution. At the same time, the rich field of mechanical methods is assigned to the periphery of the domain of geometry. In all these ways, the philosophical preoccupations of the commentators render them insensitive to the nature and motives of the older tradition of research.

Much the same applies to more recent scholarship. Examining the ancient geometry under the conception that its objectives were primarily formal, for instance, or that it subscribed to a constructivist metamathematical program or that it otherwise reflected awareness of concerns characteristic of later movements in mathematics and mathematical philosophy might yield an account more sophisticated than that proposed by the commentators from late antiquity, but is not for this more likely to be faithful to the earlier research. Indeed, the modern accounts typically attempt to build on the start provided by the commentators, assigning greater weight to a single passage by Pappus on the nature of the method of analysis or to a single short comment on the relative ordering of the techniques of construction than to the whole corpus on the analysis of problems. Once it is recognized that such statements by the commentators are founded not on the inspection of and meditation upon geometrical materials, but on paraphrases of stock positions on the related general questions discussed in the philosophical literature, one will know better how to approach the geometrical writers on the one hand and the commentators on the other in order to gain the appropriate insights into the issues which happen to be of interest. To be sure, the geometers were involved in a philosophically interesting effort, just as Herodotus' children were uttering sounds interpretable as Phrygian words. But the ones were not engaging in philosophy, just as the others were not speaking in Phrygian. Inferences on such a basis as to the relative significance of philosophical considerations within the ancient geometry will be no more valid than the Egyptians' deductions on the relative antiquity of their race.

The present survey of the ancient geometry has called attention to the prominence of the activity of investigating problems of construction. Perceiving that the problems of circle quadrature and of cube duplication did not lend themselves to solutions comparable to those more elementary problems, like the rectangle quadrature or the interpolation of a single mean proportional, effectible via familiar techniques like the use of circles and straight lines,

Hippocrates advanced the study of both, the former through his quadrature of the lunules, the latter by reducing it to the finding of two mean proportionals. Subsequent efforts by Eudoxus and his disciples developed new geometric techniques in the context of these and related problems: the generation of special curves, for instance; the implementation of the theory of the application of areas, later serving as a pillar of the theory of conic sections; the limiting techniques appropriate for effecting the measurement of curvilinear figures; and the versatile method of analysis, to become the most characteristic feature of the later problem-solving effort.

In the 3rd century B.C. this activity undergoes a split, maintained with remarkably little overlap. One branch, centering on the circle quadrature and related inquiries, with the concomitant emphasis on the use of the Eudoxean limiting methods, is dominated by the discoveries of Archimedes, who not only applied these methods successfully to new figures, like the conoids of revolution, but also effected technical refinements on the methods themselves. His findings by means of *neusis* constructions and his studies of the spiral, a model instance for the class of curves generated via mechanical motions, inspired several of the geometers in the following generation in their search for alternative constructions of the three "classical" problems.

But a second branch of the problem-solving activity during this period developed in a manner little influenced by the work of Archimedes. Spurred by the studies of locus and the applications of the theory of conics initiated by the work of Euclid and his contemporaries, geometers extended the domain of problems effected via the method of analysis. By the time of Apollonius, about a century later, the wealth of results permitted one to gain a sense of the structure of the field: which forms of conditions give rise to planar loci (circles and lines), which others to solid loci (conic sections), and accordingly, which kinds of problems are amenable to planar or solid solving methods. The surviving evidence, even in its highly imperfect condition, makes clear that these geometers aimed at exhaustion in their researches into specific classes of problems. For instance, Apollonius in one treatise secured the construction of circles answering to conditions of tangency over the whole range of possible orientations and devoted other works toward effecting the same for the conics. He explored in detail the manner of effecting forms of *neusis* constructions via the planar methods, while he and other geometers pursued the investigation of solid problems like the solution of the three- and four-line locus.

Throughout this two-century span of research, the preeminent aim is toward finding solutions to problems and discovering new properties of geometric figures. The geometers themselves stress this heuristic goal: Archimedes describes with evident pride his "mechanical method" for finding the measures of surfaces and solids and centers of gravity; Apollonius presents his compendium on the conics as an instrument for investigating locus problems; even the commentator Pappus acknowledges that the corpus of analysis is organized wholly for abetting the solution of problems. This ideal, even without such

explicit testimonies, is manifest throughout the geometrical literature. Its awesome commitment to the rigors of formal exactness in proof, that they lost sight of this heuristic objective, and it was hardly before the 17th century that geometers, impatient with the older formal style, revived this goal and set out to extend the ancient findings through their new algebraic, projective, and infinitesimal methods. Without access to techniques such as these, the ancient geometers after Apollonius could not have hoped to extend significantly the massive body of established results. It was this body, but neither the mind nor the spirit, of the older tradition which the commentators studied and transmitted, but only imperfectly understood.

NOTES TO CHAPTER 8

¹ Heath, *History of Greek Mathematics*, II, p. 358.

² Eutocius, in Apollonius, *Opera*, ed. Heiberg, II, p. 176. He expressly cites the alternative proofs and other items in his "copies" throughout the commentary.

³ Pappus, *Collection* (VII), II, p. 682; cf. Chapter 6.

⁴ For a survey of this field, see my *Evolution of the Euclidean Elements*, I, and II, I. For an account of the reconstruction of Apollonius' theory of irrationals proposed by F. Woepcke, see the edition of Pappus' *Commentary* by G. Jungé and W. Thomson, pp. 26-29, 64, 119; and T. Heath, *Euclid's Elements*, III, pp. 255-259. Pappus constructs two cases of irrational lines of the Euclidean type in *Collection* (IV), I, pp. 178-186. In the extant corpus of Archimedean writings, incommensurable magnitudes enter only once, at *Plane Equilibria* I, 7; I show how this relates to the techniques of a pre-Euclidean proportion theory in "Archimedes and the Pre-Euclidean Proportion Theory."

⁵ See Chapters 5 and 7.

⁶ One may consult Heath's *History* for brief accounts of these figures. The nontrivial significance of Eutocius' editorial work on the transmission of the *Conics* is not generally recognized; this is a major point in my "Hyperbola-Construction in *Conics* II."

⁷ Text A: *Collection* (III), I, pp. 54-56; B: (IV), I, pp. 270-272.

⁸ Here Pappus inserts a few bibliographical notes on prior studies of curves. Here text B holds a passage asserting a criterion on the proper choice of construction methods; this is translated and discussed below (see note 29).

⁹ It is difficult to say precisely what the close conformity of these two passages implies in the matter of their provenance. However, the opening of A, "The ancients say . . ." must indicate that Pappus is following a source for the content, if not the exact wording, of his remarks here. (Note this, despite the "We say . . ." in B which might, if viewed in isolation, suggest Pappus' responsibility for the view expressed.) As shown below, his "planar" and "solid" classes do indeed have precedents in the older literature, and Pappus might well have been able to draw his account from Geminus or Menelaus, both of whom wrote extensively on geometric problems and curved lines. I would thus suppose that where A and B are in agreement, Pappus is transcribing from a source.

material . . . we cannot hope . . .

¹⁰ *Collection* (VII), II, p. 652.

¹¹ As cited by Pappus, *ibid.*, (III), p. 11.

¹² *Conics*, Book I, Preface; Apollonius, *Opera*, ed. Heiberg, II, p. 178. The three examples of locus Eutocius cites are all of the planar type, one being the two-point locus discussed in Chapter 4.

¹³ *Ibid.*, p. 184.

¹⁴ *Collection* (VII), II, pp. 652, 662, 672. See the discussion of Zeuthen's proposed reconstruction in Chapter 6.

¹⁵ In *Euclidem*, pp. 394 f. Proclus uses as an illustration of a planar locus that equal parallelograms having the same base will have their upper edge lying along the same line parallel to the base; as a solid locus he cites the fact that equal parallelograms drawn in a given angle will have the opposite vertex lying along a given hyperbola.

¹⁶ *Ibid.*, p. 395. The fact a philosopher like the 3rd-century Stoic Chrysippus should find such a useful metaphor in the geometers' locus, while earlier philosophers do not mention it, suggests to me that this notion of locus was relatively recent in the 3rd century B.C.

¹⁷ *Collection* (VII), II, pp. 660-662. Here Pappus cites "the preface of *Elements*" which may refer to the general treatise on the principles of geometry attributed to him by other writers, cf. Heath, *History*, II, p. 193.

¹⁸ The terms might be translated as "restrained," "passing through," "up," respectively.

¹⁹ See note 11 above.

²⁰ Note that the latter criticism would not apply against the locus of "solid" locus maintained by Eutocius and Proclus.

²¹ *Collection* (VII), II, p. 672. Hultsch brackets this line, as does throughout his edition of the *Collection*. To the extent that his certain consistency upon the whole work, however, his effort in the compilation of these geometric materials.

²² Cf. *Conics* I, Preface (note 14) and Apollonius' writing with the locus propositions of the Aristotelian *Meteorology* (note 11, p. 636), but is notably absent from the title of Eratosthenes' place (i.e., *On Means* rather than *On Loci with Respect to*).

to base an inference, however, since such cited titles need not be original to the works. Proclus reports of the geometer Heronotimus of Colophon, shortly before the time of Euclid, that in addition to discovering many of the things in the *Elements*, he "compiled certain things on the loci" (*In Euclidem*, p. 67). This might give Heronotimus a role in the solution of problems like that in the *Meteorologica* III, 5; but again, we cannot infer from Proclus' report that the term "locus" was already then used to designate such problems. It is of interest to note the series of theorems from Euclid's *Optics*, Props. 37, 38, 44–47, of which the following is representative: "There is a place (*topos*) where, when the eye is fixed while the object seen is moved, the latter will always appear equal (in size)" (*ibid.*, 37; note the differences in the Theonine versions, Props. 41–47). These theorems all deal with the specification of *topoi* where certain optical phenomena obtain. But at the same time they yield a geometrical figure corresponding to each *topos*. It seems to me possible that contexts of this sort in optics, where *topos* retains its basic sense of "physical place," served as background for the early study of geometrical locus. This would be but one of several ways in which optical studies contributed to new developments within the geometric field in antiquity (cf. the discussion of the early theory of conics in Chapter 4; and Diocles' work on burning mirrors in Chapter 6).

²⁶ On these efforts by Menaechmus and Archimedes, see Chapters 3 and 5.

²⁷ See the résumé by Heath, *History*, II, pp. 185–189.

²⁸ This view is elaborated below.

²⁹ *Collection* (IV), I, pp. 270–272; cf. note 10 above.

³⁰ See, for instance, Heath, *History*, I, pp. 218 f., who is markedly more circumspect in this matter than most writers. The assumption of the quest for planar constructions is carefully scrutinized and convincingly discredited by A. D. Steele, "Ueber die Rolle von Zirkel und Lineal," *Quellen und Studien*, 1936, 3, pp. 289–369.

³¹ On the problem from Apollonius cited here, see Chapter 7.

³² See Chapter 5(ii) for the solid construction of Archimedes' *neusis*.

³³ See Chapter 2. Steele provides a very thorough account of this construction, arguing that the *neusis* cannot have been intended to be effected in any way other than as a *neusis*; "Zirkel und Lineal," pp. 319–322.

³⁴ See Chapter 3.

³⁵ See Chapter 4.

³⁶ See Chapter 6.

³⁷ See Chapters 5 and 6. Note that Archimedes' solution to the circle quadrature via drawing tangents to the spiral is not unrelated to the procedure of *neusis*.

³⁸ See Chapter 7.

³⁹ Such classifications can define the project of specifying the relative domains of problems solvable by the various constructing means. An impression of the wide range of results to be obtained in this area can be derived from the articles in Enriques' survey (cf. note 42).

⁴⁰ Pappus, like Hero, actually employs a *neusis*; but his method is equivalent to Diocles' use of an auxiliary curve (see Chapter 6).

⁴¹ See the statement at the end of text A cited above, and his reproduction of Hero's statement to the same effect (note 12 above).

art. 7, pp. 193–195. The irreducibility of the third-order relations for the cube duplication and the angle trisection was first shown by L. Wantzel (1837); see Liouville's *Journal*, 2, pp. 366–372.

⁴² *Collection*, I, p. 30. His lengthy and unsympathetic account of the proposed method (*ibid.*, pp. 32–48) should begin to raise doubts among those who would assume in Pappus himself a high level of geometric expertise. See the brief summary by Heath, *History*, I, pp. 268–270.

⁴³ The *Académie française* ruled in 1775 against the further consideration of contributions relating to the circle quadrature (cf. P. Beckmann, *A History of Pi*, 2nd ed., 1971, p. 173); but the first successful proof of the transcendentality of π came only with Lindemann's publication in 1882, extending an earlier result by Hermite (1873) (cf. art. 8 in Enriques, *op. cit.*). Similarly, Descartes pronounced that "it will be an error to strive in vain to wish to construct a problem by a class of lines simpler than its own nature permits" (*Géométrie*, ed. Smith, p. 157) precisely two centuries before the first specific results on the reducibility of the associated polynomials were presented (see note 42).

⁴⁴ *Collection* (III), I, p. 56.

⁴⁵ Cf. *Elements* I, 22. The implied governing construction might be "it is proposed . . ." (*probulletai*, or *proteinetai*), as sometimes supplied in discussions by Pappus, Proclus, and others, or alternatively "it is possible . . ." (*dunaton esti*). Of course, the actual construction establishes far more than the mere possibility of the construction.

⁴⁶ Cf. *Elements*, I, 21.

⁴⁷ *Ibid.*, I, 20.

⁴⁸ Cf. the locus problem in the *Meteorologica* III, 5 (Chapter 4 above).

⁴⁹ For an example of just this sort from Pappus, see Chapter 4, note 16.

⁵⁰ Pappus, *Collection* (VII), II, pp. 648 ff.; Proclus, *In Euclidem*, pp. 212, 302. Proclus recognizes an alternative sense of "porism," namely "corollary" (*ibid.*, pp. 301–303). The term derives from *porizein*, "to provide" or "to devise." An appearance of a form of the verb in the *Meteorologica* (III, 5, 376 a 15) may carry the technical sense, although another appearance in a mathematical passage in the *Ethics* (1112 b 26; discussed below) seems not to.

⁵¹ *In Euclidem*, p. 81.

⁵² This is a view proposed, among others, by Proclus and by Zeuthen. See note 77.

⁵³ Note the propositions of the form "there are places (*topoi*) such that . . ." in Euclid's *Optics* (see note 25 above). These are undeniably existential, both in sense and in form. But they are framed as *theorems* and serve as the basis for a *problem* (Prop. 48): "to find the places where the equal magnitude appears half as big, etc." Similarly, the materials in the pseudo-Euclidean *Catoptrica* culminate in a *problem* (Prop. 29): "it is possible to construct a mirror such that many faces are seen in it, some bigger, some smaller, some nearer, some further away, some reversed, some not." This serves merely to bring together the various results already established on plane, convex, and concave mirrors; it is hardly the equivalent of an existence proof.

⁵⁴ Pappus, *Collection* (IV), I, pp. 272 ff. See Chapter 7 and my "Hyperbola-

³⁶ *Conics*, Preface to Book I (ed. Heiberg, I, p. 4): "the third book (contains) many remarkable theorems useful for the syntheses and diorisms of solid loci." Similarly, the contents of Book IV are said to be "useful for the analyses of diorisms," while Book V also is "needed for obtaining a knowledge of the analysis and determination of problems, as well as for their synthesis" (cf. Heath, *Apollonius*, p. lxxiv). The same attitude is maintained by Pappus when he describes the *topos analyomenos* as "a certain special corpus . . . conducive to the finding of problems . . . and compiled as useful for this purpose alone" (*Collection*, VII, Preface, ed. Hultsch, II, p. 634).

³⁷ See Chapter 7.

³⁸ Archimedes does without any such existence proof in the area theorem of *Dimension of the Circle*, Prop. 1; the omission was criticized by some, but defended by Eutocius (*Archimedes*, *Opera*, III, p. 230; see discussion below). Further examples relating to the alleged existential function of problems appear in chapter 3.

³⁹ Note that it was the theorematic form of Postulates 4 and 5 which encouraged ancient geometers like Apollonius and Ptolemy, respectively, to seek proofs for them; cf. Proclus, *In Euclidem*, pp. 188 ff, 191 ff, 365 ff. Proclus asserts flatly that the "postulate" on parallels is a theorem (*ibid.*, p. 191).

⁴⁰ *Collection* (III), I, p. 30; cf. (VII), II, pp. 648 ff. Note that the notion of displacing all problems by theorems or *vice versa* receives no support from the technical literature, where both forms appear and serve clearly marked purposes. Comparable distinctions in the dialectical roles of problems and theorems (e.g., the distinction between *problemata* and *protaseis* in debating contexts, *Topics* I, 4, 10, 11) or in the different priority of terms in the order of knowing and the order of proving (e.g. *Posterior Analytics* I, 2) may suggest a loose connection between these discussions on geometry and the Aristotelian epistemology.

⁴¹ *Republic* 527; cf. 510d and *Euthydemus* 290c.

⁴² *In Euclidem*, pp. 77 f. On Amphinomus, see Chapter 3.

⁴³ See the discussion of the *Platonicus* fragment, Chapter 2.

⁴⁴ *In Euclidem*, p. 243.

⁴⁵ *Ibid.*, p. 78.

⁴⁶ See Aristotle's *Metaphysics* Z.10, 1036 a 9; and the discussion by Heath, *Mathematics in Aristotle*, pp. 213–216, 224–226.

⁴⁷ Proclus, *In Euclidem*, pp. 241 f; Pappus, *Collection* (VIII), III, p. 1026; Simplicius, *In Physica*, ed. Diels, p. 60. The identity of this curve is not known; Tannery conjectures it was the cycloid, but I have argued that it may be the cylindrical helix (see my "Archimedes and the Spirals," pp. 73 f).

⁴⁸ *In Euclidem*, p. 242.

⁴⁹ *Ibid.*, pp. 80 f.

⁵⁰ Here Zenodotus is described as "in the succession of Oenopides" and "among the disciples of Andron." While Oenopides would be the 5th-century geometer-astronomer, this helps little in placing Zenodotus. Given the later reference to Posidonius in this same passage, an identification with a Stoic writer of this name from the 2nd century B.C. might be indicated (cf. *Pauly Wissowa*, "Zenodotus (4) and (6)"), but this is entirely conjectural. So also is the degree to which such late writers reflect the actual ideas of the earlier writers with whom they are associated. K. von Fritz has observed that Zen-

odotus' observation about problems (i.e., that they seek "that on the assumption of which something is," *tinon onto ti esti*) has to do with the *analysis* of problems ("Oenopides" in *Pauly Wissowa*, reprinted in his *Schriften zur griechischen Logik*, Stuttgart, 1978, 2, pp. 151 f). Note, however, that the implied conception of analysis accords with the view of Pappus, following the Aristotelian manner, that is, as a search for suitable *antecedents*, rather than as an investigation of consequences (see discussion below). Of course, one can hardly hope to ascribe to Oenopides a role in the introduction of the analytic method on the basis of this evidence.

⁵¹ This is surely Posidonius, the distinguished Stoic teacher of the late 2nd century B.C.

⁵² The manuscripts here insert *problēma* which Morrow understandably rejects as an interpolation, since it violates the sense of the passage (cf. his *Proclus A Commentary* . . . p. 66n).

⁵³ *In Euclidem*, pp. 233 f.

⁵⁴ A view minimizing the formal intent underlying Euclid's geometry is argued by A. Seidenberg in "Did Euclid's Elements, Book I, Develop Geometry Axiomatically?" *Archive for History of Exact Sciences*, 14, 1975, pp. 263–295.

⁵⁵ Oddly, Proclus can here cite some Epicureans who criticize proving this result at all, given its perfectly obvious character (*ibid.*, p. 322).

⁵⁶ See Heath, *Euclid's Elements*, I, pp. 293 f. Note that if the analysis of this problem had been given first, the necessity of this condition and its role within the synthesis would have been clear. It would so appear that Euclid's formal difficulties in cases like this follow from his decision to present only the syntheses of the problems.

⁵⁷ I believe that the *locus classicus* for the thesis that the ancient geometric constructions were meant to serve as proofs of existence is Zeuthen's "Die geometrische Construction als 'Existenzbeweis' in der antiken Geometrie," *Mathematische Annalen*, 47, 1896, pp. 222–228. His case rests primarily on Euclidean materials, although he draws from more advanced work as well. But when he supposes, for instance, that the Greeks had to seek a construction of the conics (e.g., as sections of a cone) because they would not admit mechanical or pointwise methods, he commits a plain error (cf. the discussions of Menaechnus and Diocles in Chapters 3 and 6). It is thus hard to understand how the thesis has gained such credence. To be sure, one can find telling instances where problems enter to justify the introduction of terms necessary for subsequent theorems. An instance not cited by Zeuthen is *Elements* XII, 16, 17: two problems assumed in steps in the proof of Prop. 18 (that spheres are as the cubes of their diameters). But we have seen other instances where existence is established via *theorems*, subordinate to problems (cf. note 54 above), as well as many cases where the problems of construction have standing in their own right, without reference to associated theorems (cf. note 56 and Chapter 3). We can only conclude that the ancients did not distinguish proofs of existence from other geometric propositions in the way modern mathematicians do; that apart from diorisms (which are by their nature existential in force) no special format was reserved for questions of existence as they arose, these being handled now as theorems, now as problems, but often only as tacit assumptions. For a nicely balanced view on the relation of ancient geometric constructions and existence proofs, see I. Mueller, *Philosophy of Mathematics* . . . pp. 15 f, 27–29.

⁵⁸ For the scholia, see Euclid, *Opera*, ed. Heiberg, IV, pp. 364 ff. The commentary by al-Nairizi (Annariti) survives both in Arabic and in its Latin translation by Gerard

of Cremona. I have consulted the latter in the edition by M. Curtze, in *Euclid, Opera*, ed. Heiberg, IX; for references to the former, see F. Sezgin, *Geschichte des arabischen Schrifttums*, V, pp. 283–285.

⁷⁹ See the contributions cited by J. Hintikka and U. Remes, *The Method of Analysis*, 1974, Ch. ii. The authors have compiled an extensive survey of ancient comments on analysis in their Ch. viii.

⁸⁰ *Collection* (VII), II, p. 634.

⁸¹ *Ibid.*

⁸² *Euclid, Opera*, IV, p. 364.

⁸³ I follow the Latin of Gerard in *ibid.*, IX, p. 89. He seems here to have made a valiant attempt to render a difficult passage. The Arabic (Leiden, Or. 399, f. 25v, lin. 10–12) can be given thus:

As for analysis (*tahlil*), it is when any question has been proposed to us, we say (that) we take it down (*manziluhā*) (to) the degree (*manzila*) of the thing sought (*maṭlūb*), that it is found (*innahu māwjud*), then we break it up (*nafudduhu*) to something whose proof has already preceded, so that since it is manifest to us, we say that what was sought (*maṭlūb*) has now been found (*wujida*) through analysis.

Hintikka and Remes cite a translation made for them by T. Harviainen from the Arabic: "we suppose the thing sought as being (*mawjūd*), then we resolve it to something already proved." (*Method of Analysis*, p. 93) They thus wish to read *innahu māwjud* as a version of Greek *hōs on*. But in context, *mawjūd* must surely here mean "found," as the verbal form *wujida* clearly does later in the passage. The underlying Greek would thus be *hōs zētoumenon*, that is, "as if found." The Arabic translator has lost the nuance of *hōs* (present in the parallel Greek passages cited: *hōs homologoumenon*). Consequently, Gerard renders as *que est inventa* what, in light of the presumed Greek, would more accurately be *quasi inventa*. As for terminology, "analysis" becomes *tahlil* ("dissolutio"), "synthesis" is *tarkib* ("compositio").

⁸⁴ This looseness of the senses of *akolouthēin* is emphasized by Hintikka and Remes, *op. cit.*, pp. 13 ff; they opt for the rendering "concomitants."

⁸⁵ Note that the scholiast speaks of the thing admitted as being "true," and so would seem to have in mind a theorematized context of analysis, like that described later by Pappus. Hero's term "known" could apply equally well to the analysis of problems and that of theorems. The phrase "up to something admitted as true" here only poorly suits the context, and is likely to be a scribal error due to the appearance of the same phrase in the preceding sentence (see note 82). Some manuscripts here have an alternative phrase, "up to the ending up or taking of what is sought" (cf. Heiberg, *Euclidis Opera*, IV, p. 366n).

⁸⁶ It seems difficult to assign to "antecedents" and "consequents" here their usual logical senses and yet to remain compatible with Pappus' notion of the order of analysis given before; for there the terms found were logical antecedents, even though they happened to be discovered later in the analysis. Of course, in the usual presentation of an analysis, terms are derived as logical consequences, so that a reversal of the logical order does indeed have to be secured in producing the synthesis.

⁸⁷ In the *Metrica*, Hero presents many examples of "analyses" of an interesting variant sort. In these, the objective is to derive a computational rule or procedure by

means of what is called an "analysis" and then in the following "synthesis" to work out a solution in particular numbers via the derived rule. In *Metrica* III, 4–14, 20–22, these are all problems in the division of figures, and one can appreciate the appropriateness of the analysis/synthesis distinction. But in *Metrica* I, 8–25 and II, 6–9, 13, the rules all involve expressions for the areas and volumes of figures. These are all stated as *problems* (e.g., "to find the area of the regular heptagon" in I, 20), but they have no analytic hypothesis of the sort which dominates the analysis of a problem. On the other hand, these analyses do start from the whole figure and work through to a relation of a computable sort pertaining to the givens (e.g., the given length of 10 for the side of the heptagon), while the syntheses begin with the givens and calculate with these in accordance with the derived relation to produce what is sought. Doubtless, this format was adapted from the analysis of problems in the more advanced geometry. Hero's version may be based on a prior tradition following this same format; he depends, for instance, on values drawn from "the books on chords" (a reference to Hipparchus? cf. *Opera*, ed. Schöne, p. 58 n), but his heading to the section of analyses and syntheses is ambiguous (*ibid.*, p. 16). At any rate, Pappus seems to know of this format; for he invites the reader to work out numerically a value for the cube duplication "in accordance with the analysis" through consultation of Ptolemy's table of chords (*Collection*, I, p. 48).

⁸⁸ *Collection* (VII), II, pp. 708–710; cf. Heath, *History*, II, pp. 406 f.

⁸⁹ See, for instance, analyses of the problems on the inscription of the regular polyhedra in *Collection* III, Prop. 54 ff. The same appears in Dioctes' analysis of Archimedes' solid problem (cf. Toomer, *Dioctes*, p. 86). In some instances, the synthesis might merely be omitted without comment, as in Pappus' treatment of the analyses of the "solid *neusis*" used by Archimedes (*Collection* (IV), I, Prop. 42 f). It is interesting, however, that in most such cases the commentators Eutocius and Pappus will supply the missing synthesis, or at least fill in a sketch of it.

⁹⁰ It is of course possible that Pappus himself is responsible for this lemma. It would then merely be a reflection of his own view of the nature of analysis, rather than evidence from the more general technical field.

⁹¹ *Nicomachean Ethics* III.3, 1112 b 15–27.

⁹² The passage is well noted by Hintikka and Remes (*op. cit.*, pp. 85 f) who draw attention to the remarkable conformity in terminology between it and the account of analysis by Pappus. But they seem reluctant to admit that Pappus might have referred to it as a model for his own statement.

⁹³ This observation is made by Hintikka and Remes, *op. cit.*, p. 86.

⁹⁴ *Collection* (III), I, p. 86 (i.e., that the geometric mean is a cause of harmony in all things; this would appear to relate to *Timaeus* 31–32, as noted by Hultsch, *ibid.*, p. 87n).

⁹⁵ These are noted by Thomson in his introduction to the *Commentary of Pappus* . . . , pp. 51–57.

⁹⁶ An extreme example of the mathematical commentary conceived within the framework of Platonist doctrines is that by Proclus on Euclid's Book I; see the introductory remarks by Morrow, *op. cit.*, pp. xxiv–xlili.

⁹⁷ *Collection* (VII), II, pp. 634–636.

⁹⁸ Note how Pappus' view here of the two possible outcomes of the analysis recalls Aristotle's observation to the same effect in the *Ethics* passage cited above.

⁹⁹ Cf. Hultsch in *Collection*, II, p. 636n. This line is deleted without comment in the versions of Pappus' account given by Heath (*History*, II, p. 401) and by I. Thomas (*Greek Mathematical Works*, II, p. 598).

¹⁰⁰ *Op. cit.*, Chs. iv–vi.

¹⁰¹ Hintikka and Remes (*ibid.*, Ch. iv) insist that the theoretic form of analysis be taken quite seriously and attack in sharp terms any view that the heuristic processes underlying problem solving and theorem proving are somehow distinct (*ibid.*, p. 47, criticizing M. Mahoney, "Another Look at Greek Geometrical Analysis," *Archive for History of Exact Sciences*, 1968–69, 5, pp. 319–348, especially p. 328). But surely two different things are at issue here. Let us of course accept the authors' findings on the logical description of the problems and theorems, to the effect that the description is the same in either case. Nevertheless, the analysis of a theorem in the sense they require must begin with the hypothesis that the result claimed in the theorem is true. How is that result arrived at? In the pedagogical context, the teacher might present a claimed theorem to the student and ask him to prove (or disprove) it; in this case, the student might well employ an analytic procedure to assist him in working out a proof. Otherwise, the analytic hypothesis must be a matter of mere conjecture. One can well doubt whether the ancients approached the finding of proofs of theorems *as theorems* in such a context-free manner. Furthermore, in cases like Archimedes' theorem on the area of the circle (that it equals one-half the radius times the circumference), the proof results from hypothesizing that the claimed result is *false*. It is hard to see the value, from the historical point of view, of assimilating the techniques used for investigating such theorems with the analytic method used for problems, even if this proves to be rewarding from the logical point of view.

¹⁰² *Collection* (IV), I, p. 186. This example is discussed in detail in Hintikka and Remes, *op. cit.*, Ch. iii. The lemma which closes *Collection* VII (II, pp. 1016–20) is also an attempt toward the analysis of the theorem, although the text has been severely garbled. Other than these, I know of no extant analyses of theorems, save for those of the trivial sort mentioned earlier (see note 78 and note 87).

¹⁰³ Cf. *Prior Analytics* I, 23, 41 a 21 ff.

¹⁰⁴ Cf. Proclus, *In Euclidem*, pp. 212 f.

¹⁰⁵ Pappus is not alone in proposing the theoretic form of analysis; there are comparable statements from Proclus (cf. *In Euclidem*, p. 42; if indeed the "analytic faculty," or *analytikê dynamis*, mentioned there refers to the geometric analysis) and from his disciple Marinus (*In Euclidis Data*). The latter writes, for instance, that

the knowledge of the *data* is most necessary for the purposes of the so-called analytic corpus (*topos analyomenos*). . . analysis is the finding of proof, it is compiled for us for finding the proof of the similar things, and acquiring the analytic faculty is greater than having many proofs of things in part. (Menge, *Euclidis Opera*, VI, pp. 252–254)

In citing these passages, Hultsch notes their resemblance to Pappus' account of analysis, the verbal reminiscences hardly assignable to mere coincidence (*Collection*, III, pp. 1275 f.). Hultsch infers, not the later writers' use of Pappus, but their common dependence on a much older account of analysis. Since Euclid, Apollonius, and Aristaeus were the principal authors of this corpus, he reasons, one of them must have been responsible for this alleged account, and doubtless Euclid himself was the one, since he was the oldest among them. Apparently, one must suppose that geometers could not apply the method

of analysis without also preparing and handing down a form of philosophical statement on its nature; and apparently, a later commentator like Geminus was incapable of composing such a statement without an explicit model in hand deriving ultimately from Euclid or his disciples.

¹⁰⁶ See my account of this theorem in *Evolution of the Euclidean Elements*, Ch. VIII/IV, especially pp. 279 ff. Proposition XIII, 1 is also used in the proof of XIII, 6; but the latter has been convincingly argued to be an interpolation (cf. Heath, *Euclid's Elements*, III, p. 451).

¹⁰⁷ Even with this modification, the analysis in this case would be unnecessary, since the claimed result is an obvious feature of the construction of this section of the line in *Elements*, II, 11. That is, x is produced by taking the line $x + z$ equal to the hypotenuse of a right triangle whose legs are z and $2z (= x + y)$; see Chapter 3. One is thus led to suspect that XIII, 1, like XIII, 6, is an interpolation—whence perhaps the entire series of Lemmas 1–6 may be a later addition to the body of constructions and theorems of that book.

¹⁰⁸ Proclus, *In Euclidem*, p. 242. See note 67 for other references to Carpus.

¹⁰⁹ Proclus, *ibid.* For further references to this debate, see notes 60ff.

¹¹⁰ One may cite Eutocius' removal of the analysis in his version of the hyperbola construction (see Chapter 7, note 33) and Pappus' stated preference for the synthetic mode by virtue of its greater conciseness in comparison with the analytic mode (*Collection*, I, pp. 410–412). One notes that the angle trisections presented by Thābit ibn Qurra and Ahmad ibn Mūsā (see my "Transmission of Geometry from Greek into Arabic") eliminate the analyses Pappus presents (cf. *Collection*, I, pp. 270 ff.). Other examples of the commentators' preference for syntheses are cited in note 89 above.

¹¹¹ *Collection* (VII), II, pp. 650–652.

¹¹² An alternative translation: "only that this is what is sought," seems possible, but does not conform to the sense of the passage.

¹¹³ *Collection* (III), I, p. 30.

¹¹⁴ *Ibid.*, (IV), I, p. 284.

¹¹⁵ For references to the accounts by ps.-Alexander and by Alexander as reported by Philoponus, see Chapter 3.

¹¹⁶ Philoponus, *In Analytica Posteriora*, ed. M. Wallies (CAG 13), p. 112; see Chapter 3.

¹¹⁷ Commentary on *Dimension of the Circle*, Prop. 1, in Archimedes, *Opera*, III, p. 230.

¹¹⁸ See Chapter 3.

¹¹⁹ *In Categorias*, ed. A. Busse (CAG 4, Pt. 4, 1895), p. 75.

¹²⁰ Philoponus, *op. cit.*, p. 120; Simplicius, *In Categorias*, ed. K. Kalbfleisch (CAG 8, 1907), p. 192.

¹²¹ Cf. Simplicius, *loc. cit.* and also *In Physica*, ed. Diels, p. 60.

¹²² Commentary on the *Data* in Euclid, *Opera* VI, ed. Menge, p. 240.

¹²³ See the survey in my *Evolution of the Euclidean Elements*, e.g., Ch. IX.

¹²⁴ *Opera* III, ed. Schöne, p. 172; for an account, see Heath, *History*, II, pp. 339 f. Hero's construction here seems related to another problem in the sectioning of a circle in Euclid's *Division of Figures*, Prop. 29 (cf. Heath, *ibid.*, I, p. 429). Note that in both

cases, although the problem seeks a part which is *one-third* the circle, the method is clearly a general one.

¹²⁵ This conjecture is made by Maimonides in his comment to *Mishnah Eruvin* I, 5 (for which reference I thank Daniel Lehmann, Department of Mathematics, University of Southern California). Maurolico bases his own conjecture on the isoperimetric property of the circle; cf. M. Clagett, *Archimedes in the Middle Ages*, III, pp. 779n, 782n. For references to Stifel and Gregory, see Hobson, *op. cit.*, p. 31. Note that Descartes seems to be of the same view when he observes that the motions which yield the spiral, the quadratrix, and other such curves, that is, a circular motion synchronized with a linear motion, do not have a ratio which is exactly measurable (*Géométrie*, II, ed. Smith, p. 45). He may well have in mind the similar remark made by Sporus, as reported by Pappus (*Collection*, I, p. 254), that the synchronization requires that this ratio be known, yet the object of the use of these curves is toward finding that ratio. On Lambert's proof of the irrationality of π (1761/68), see Hobson, *op. cit.*, pp. 43 ff; and Enriques, *op. cit.*, art. 8.

¹²⁶ Eutocius, in Archimedes, *Opera*, III, p. 260.

¹²⁷ *Ibid.*, p. 232.

¹²⁸ In *Physica* VII.4, ed. Diels, pp. 1082 f. Simplicius also observes here that the circle quadrature "has not yet been found; but even if it might seem at this time to have been found, it is only through certain disputed hypotheses." One seems to hear an echo of this view in a remark by the Banū Mūsā: that one can compute an approximation to the value of the ratio of the circumference and diameter as closely as desired via the given (Archimedean) method, while "no one has to this day found anything beyond what has appeared to us" (*Verba filiorum*, Prop. 6; Clagett, *Archimedes*, I, p. 264).

¹²⁹ In *Physica*, p. 60.

¹³⁰ In *Categorias*, p. 192.

¹³¹ *Collection* (IV), I, pp. 252 ff; cf. note 125 above.

¹³² Cf. the survey of angle trisections by al-Sijzī (cited in Woepcke, *Omar*, p. 120), where the "mobile" manner of the "ancient" (i.e., Archimedean) construction is criticized. Most forms of the Arabic terms, *thabī* ("fixed") and *bayyin* ("manifest"), are indistinguishable in unpointed text. In the account of Nicomedes' cube duplication (see chapter 6) by Abū Ja'far Muḥ. b. al-Ḥusain it is the ancient "method of instruments" which is replaced by a "geometric" procedure, the latter being the same as Pappus' use of conics for effecting the *neusis* for the angle trisection; cf. Sezgin, *op. cit.*, p. 306.

¹³³ Cf. *Vita Marcelli*, xiv and the references cited in Chapter 2.

¹³⁴ See the citation by Pappus in *Collection* (VIII), III, p. 1026.

¹³⁵ See Chapter 6.

¹³⁶ On the cube duplication, see the references in Chapter 5, note 115. Note his inclusion of practical methods for the measurement of figures in *Metrica* I, 39 and II, 20; his remarks on the practical interest of measures of the torus and of the sectioned cylinder (II, 13 and 16); his devotion of the entire writing *On the Dioptra* to the practical use of this angle-sighting device in surveying.

¹³⁷ *Collection* (VIII), III, pp. 1072–74. Hultsch's translation of this as *proble-mata... sine demonstratione geometrica solvuntur* (that is, "problems solved without geometric demonstration"; *ibid.*, p. 1075) cannot be right.

¹³⁸ See the remarks on this problem in Chapter 7.

¹³⁹ This is a point made with considerable force by Descartes in the opening section of Book II of his *Géométrie*.

¹⁴⁰ See, for instance, Euclid's Book XI, Def. 18 and Apollonius' Book I, Def. 1. In their comments on this part of Apollonius, neither Eutocius nor Pappus indicates knowledge of any ancient attempt to formalize these conceptions.

¹⁴¹ On these *neuses*, see note 38 in Chapter 7.

¹⁴² See, respectively, Chapters 2(ii); and 5(ii); and 6 (note 106).

¹⁴³ For the text, see Philoponus, *In Ana. Post.*, p. 105n; reproduced by Heiberg in Apollonius, *Opera*, II, p. 106.

¹⁴⁴ See Chapter 7.

¹⁴⁵ *Histories* II, 2.

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